1.1-1.2

Problem (1.1.5)**.** *Let Y be a modification of X, and suppose both processes have a.s. right-continuous sample paths. Then X, Y are indistinguishable.*

Proof. Let $N = \mathbb{Q}^+ \subseteq \mathbb{R}^+$, which is a dense subset, then $\mathbb{P}[X_{t_n} = Y_{t_n}, t_n \in A] = 1$. Call the set in the argument Ω. Let $\omega \in \Omega$, and $t \in \mathbb{R}^+$ be arbitrary. Then there exists $\{t_n\} \subseteq N$ such that $X_t(\omega) = \lim_{t_n \to t^+} X_{t_n}(\omega) = \lim_{t_n \to t^+} Y(\omega)_{t_n} = Y(\omega)_t$. Hence $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$ for all $\omega \in \Omega$. QED. □

Problem (1.1.7)**.** *let X be a process, every sample path of which is RCLL (i.e., right-continuous with finite left-hand limit). Let A be the event that X is continuous on $[0, t_n)$. Show that $A \in \mathcal{F}^X_{t_0}$.*

Proof. Let A be the above set, question is what characterizes continuity?

$$\{X_t \text{ continuous on } [0, t_0)\} = \bigcap_{n \in \mathbb{N}} \bigcup_{\delta \in \mathbb{Q}^+} \bigcap_{x,y \in [0,t_0) \cap \mathbb{Q}^+, |x-y| < \delta} \{|X_x - X_y| < \frac{1}{n}\}$$

where $\{|X_x - X_y| < \frac{1}{n}\} \in \mathcal{F}_{t_0}$ for $x, y < t_0$ and we have only countable union and intersections above, so $A \in \mathcal{F}_{t_0}$. □

Problem (1.1.8)**.** *Let X be a process whose sample paths are RCLL a.s. and let A be the event that X is continuous on $[0, t_0)$. Show that A can fail to be in $\mathcal{F}^X_{t_0}$, but if the $\mathcal{F}_t$ is that $\mathcal{F}^X_t \subseteq \mathcal{F}_t$ and $\mathcal{F}_t$ is complete under the probability measure $\mathbb{P}$, then $A \in \mathcal{F}_{t_0}$.*

Proof. Let B be the set for which $X_t(\omega)$ is not right continuous or has no left limit. Then $A^c = \{X_{t^-} \neq X_t\} \cap \{X_{t^-}$ exists$\} \cup \{X_t$ not right continuous for some $t\} \cup \{X_{t^-}$ does not exist for some $t\}$, where $\{X_{t^-}$ exists$\}, \{X_t$ not right continuous for some $t\}, \{X_{t^-}$ does not exist for some $t\}$ where $t < t_0$ are all null sets since they are subsets of B. However, we do not konw if they are measurable. However, if $\mathcal{F}^X_{t_0}$ is complete under $\mathbb{P}$, the the arguments of the previous problem would work exactly same with those null sets in $\mathcal{F}^X_{t_0}$. □

Problem (1.1.10)**.** *Let X be a process with every sample path LCRL (i.e. left-continuous on $(0, \infty)$ with right hand limit on $[0, \infty)$), and let A be the event that X is continuous on $[0, t_0]$. Let X be adapted to a right-continuous filtration $\{\mathcal{F}_t\}$. Show that $A \in \mathcal{F}_{t_0}$.*

Proof. Not sure why we need the right continuity, but anayway, A is the set where $X_t(\omega)$ is uniformly continuous on $[0, t_0]$:

$$A = \bigcap_{n \in \mathbb{N}} \bigcup_{\delta \in \mathbb{Q}^+} \bigcap_{t_1, t_2 \in [0,t_0] \cap \mathbb{Q}^+, |t_1 - t_2| < \delta} \{|X_{t_1} - X_{t_2}| < \frac{1}{n}\}$$

□

Problem (1.1.16)**.** *If the process X is measurable and the random time T is finite, then the function X_T is a random variable.*

Proof. X measurable means $X_t(\omega) \in \mathcal{F} \times \mathcal{B}(\mathbb{R}^+)$. $T \leq a < \infty$ insures X_T is defined for all ω, then composition of measurable function is measurable. □

Problem (1.1.17)**.** *Let X be a measurable process and T a random time. Show that the collection of all sets of of the form $\{X_T \in A\}$ where $A \in \mathcal{B}(\mathbb{R})$ together with the set $\{T = \infty\}$, forms a sub-σ-field of $\mathcal{F}$. We call this σ-field generated by X_T.*

Proof. First, $\{T = \infty\} \in \mathcal{F}$ since T takes values on the extended positive real line, then by previous problem, we are done. □

Problem (1.2.2)**.** *Let X be a stochastic process with T a stopping time of $\{\mathcal{F}^X_t\}$. Suppose for some $\omega, \omega' \in \Omega$, we have $X_t(\omega) = X_t(\omega')$ for all $t \in [0, T(\omega)] \cap [0, \infty)$. Show that $T(\omega) = T(\omega')$.*

Proof. Hint from: https://math.stackexchange.com/questions/625580/equality-of-value-implies-equality-of-stopping-time
First show $\omega' \in \{T \leq T(\omega)\}$: The collection $\mathcal{C}$ of subsets $A \subseteq \Omega$ such that $1_A(\omega) = 1_A(\omega')$ forms a σ-field. Suppose $X_t(\omega) = X_t(\omega')$ for all $t \in [0, T(\omega)]$, and let $B \in \mathcal{B}(\mathbb{R})$, then $X_t^{-1}(B) \in \mathcal{C}$ for all $t \in [0, T(\omega)]$, hence $\sigma(X_t) \in \mathcal{C}$, hence $\sigma(X_t; 0 \leq t \leq T(\omega)) = \mathcal{F}^X_{T(\omega)} \subseteq \mathcal{C}$. Therefore, $\omega' \in \{T \leq T(\omega)\} \in \mathcal{F}^X_{T(\omega)}$. Then the same arugument show that other inclusion, hence we are done. □

For next two problem:
Let X be stochastic process with right-continuous paths, which is adapted to a filtration $\{\mathcal{F}_t\}$. Consider a subset $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ of the state space of the process, and define the *hitting time*:

$$H_\Gamma(\omega) = \inf\{t \geq 0 : X_t(\omega) \in \Gamma\}$$

Problem (1.2.6). *If Γ is open, show that H_Γ is an optional time.*

Proof. For simplicity, call H_Γ to be T. By definition of the hitting time, if $T(\omega) = s < t$, then $\forall \delta \geq 0, A_\delta = \{X_t(\omega) : s \leq t \leq s+\delta\} \cap \Gamma \neq \emptyset$, let $X_{t_\delta}(\omega) \in A_\delta$. Since Γ is open, for all $\epsilon > 0$, we have $B = B(X_{t_\delta}(\omega), \epsilon) \subseteq \Gamma$. By right continuity, there exists $\gamma > 0$ such that $\{X_t : t_\delta \leq t \leq t_\delta + \gamma\} \subseteq B$, where γ can be arbitrarily small. Therefore, for all $\delta > 0$, there exists $z \in [s, s+\delta) \cap \mathbb{Q}^+$ such that $X_z(\omega) \in \Gamma$, in other words, z can be taken to be less than t. Therefore, we have the following

$$\{T < t\} = \cup_{s \in [0,t) \cap \mathbb{Q}^+} \{X_s \in \Gamma\}$$

since the $\supset$ is obvious and $\subseteq$ is proven above, then $\{X_s \in \Gamma\} \in \mathcal{F}_{t_0}$, so is the countable union, hence we are done. □

Problem (1.2.7). *If the set Γ is closed and the sample paths of the process X are continuous, then H_Γ is a stopping time.*

Proof. Let $\Gamma_n = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Gamma) < \frac{1}{n}\}$, which is open, and T_n be the hitting time of Γ_n which is an optional time from the previous problem. By continuity, $T_n \to H_\Gamma$ pointwise from below.

Can we say $\{H_\Gamma \leq t\} = \lim_{n\to\infty}\{T_n < t\}$? No, at least not at this point.

Note if $H_\Gamma(\omega) = 0 \iff T_n(\omega) = 0 \forall n$.

For $t \neq 0$ we have $H_\Gamma(\omega) \leq t \Rightarrow \lim_{n\to\infty} T_n(\omega) \leq t \iff T_n(\omega) < t$ for all $n \geq 1$. Therefore, for $t > 0$, we have

$$\{T \leq t\} = \bigcap_{n\in\mathbb{N}} \{T_n < t\} \in \mathcal{F}_t$$

The answer key says $T_n \to T$ is not obvious, so let's show it. Let's consider any particular path, so everything above is fixed and not random. Since T_n is bounded above and nondecreasing, since it converges to some H. Then $X_H \in \bigcap_{n\in\mathbb{N}} \Gamma_n = \Gamma$. Now, if $H < T$, then we'd have a contradiction. □

Problem (1.2.10). *Let T, S be optional times; then $T + S$ is optional. It is a stopping time if one of the following conditions holds:*

$$(i) T > 0, S > 0;$$
$$(ii) T > 0, S \text{ is a stopping time}$$

Proof. For the first part, we can do the following decomposition:

$$\{S+T \geq t\} = \{S \geq 0, T \geq t\} \cup \{T \geq 0, S \geq t\} \cup \{0 \leq S \leq t, S+T \geq t\} \cup \{0 \leq S \leq t, S+T \geq t\}$$

The first two sets are in $\mathcal{F}_t$, and we only need to show one of the fourth and third set is in $\mathcal{F}_t$. So consider the third set

$$\begin{aligned}
\{0 \leq S \leq t, S+T \geq t\} &= \bigcap_{n\in\mathbb{N}} \{0 \leq S \leq t + \frac{1}{n}, S+T \geq t\} \\
&= \bigcup_{r \in \{0\} \cup \mathbb{Q}^+ \cap [0,t]} \bigcap_{n\in\mathbb{N}} \{r \leq S < t + \frac{1}{n}, S+T \geq t\} \\
&= \bigcup_{r \in \{0\} \cup \mathbb{Q}^+ \cap [0,t]} \bigcap_{n\in\mathbb{N}} \{r \leq S < t + \frac{1}{n}, s+T \geq t\} \\
&\in \mathcal{F}_t
\end{aligned}$$

Therefore, it is a optional time.

(i) From **Lemma 2.9** we have

$$\begin{aligned}
\{T+S > t\} &= \{T = 0, S > t\} \cup \{T > t, S = 0\} \cup \{T \geq t, S > 0\} \cup \{0 < T < t, T+S > t\} \\
&= \{T \geq t\} \cup \{T < t, T+S > t\}
\end{aligned}$$

Where the first set is already in $\mathcal{F}_t$, so let's consider the second one:

$$\begin{aligned}
\{0 < T < t, S+T > t\} &= \bigcup_{r \in Q^+ \cap (0,t)} \{r \leq T < t, S+r > t\} \\
&= \bigcup_{r \in Q^+ \cap (0,t)} \bigcup_{n\in\mathbb{N}} \{r \leq T < t, S+r \geq t + \frac{1}{n}\}
\end{aligned}$$

which is an element of $\mathcal{F}_t$.

(ii) Now assume $T > 0$ and S is stopping time, we stil use the same decomposition:

$$\begin{aligned}\{T+S>t\} &= \{T=0,S>t\}\cup\{T>t,S=0\}\cup\{T\geq t,S>0\}\cup\{0<T<t,T+S>t\}\\ &= \{T>t,S=0\}\cup\{T\geq t\}\cup\{T<t,T+S>t\}\end{aligned}$$

first and second sets are in $\mathcal{F}_t$. We can rewrite the last set as

$$\{0<T<t,S+T>t\} = \bigcup_{r\in Q^+\cap(0,t)}\{r\leq T<t,S+r>t\}$$

which is in $\mathcal{F}_t$. □

Problem (1.2.13). *Verify that $\mathcal{F}_T$ is actually a σ-field and T is $\mathcal{F}_T$ measurable. Show that if $T(\omega)=t$ for some constant $t\geq 0$ for all $\omega\in\Omega$, then $\mathcal{F}_T=\mathcal{F}_t$.*

Proof. Recall that $\mathcal{F}_T=\{A\in\mathcal{F}:A\cap\{T\leq t\}\in\mathcal{F}_t\}$. Obviously, it is closed under countable internsections: $\bigcap_{n\in\mathbb{N}}A_n\cap\{T\leq t\}=\bigcap_{n\in\mathbb{N}}(A_n\cap\{T\leq t\})\in\mathcal{F}_t$. Now let $A\in\mathcal{F}_T$, consider

$$\begin{aligned}A^c\cap\{T\leq t\} &= (A^c\cap\{T\leq t\})\cup(\{T\leq t\}\cap\{T>t\})\\ &= (A^c\cap\{T>t\})\cap\{T\leq t\}\\ &= (A\cap\{T\leq t\})^c\cap\{T\leq t\}\end{aligned}$$

Now, $\sigma(T)$ is generated by $T^{-1}((-\infty,t])\in\mathcal{F}_T$, hence the generated σ algebra is also a subset of $\mathcal{F}_T$.

Finally, let $T\equiv t$, then $\{T\leq t\}=\Omega$, and if $A\in\mathcal{F}_T$, then $A\cap\Omega=A\in\mathcal{F}_t$ so $\mathcal{F}_T\subseteq\mathcal{F}_t$. For the other direction, if $A\in\mathcal{F}_t$, then $A\cap\{T\leq t\}=A\in\mathcal{F}_t$, so we are done. □

Excercise (1.2.13). *Let T be a stopping time and S be a random time such that $S\geq T$ on Ω. If S is $\mathcal{F}_T$ measurable, then S is also a stopping time.*

Proof. Here we need to show $\{S\leq t\}\in\mathcal{F}_t$ for all $t\geq 0$, however, we have $\{S\leq t\}\cap\{T\leq t\}\in\mathcal{F}_t$, but $\{S\leq t\}\subseteq\{T\leq t\}$, so we are done. □

Problem (1.2.17). *Let T,S be stopping times and Z be an integrable random variable. We have*

$$\begin{aligned}&(i)E[Z|\mathcal{F}_T]=E[Z|\mathcal{F}_{S\wedge T}],\ P\text{-a.s. on }\{T\leq S\}\\ &(ii)E[E(Z|\mathcal{F}_T)|\mathcal{F}_S]=E[Z|\mathcal{F}_{T\wedge S}],\ P\text{-a.s.}\end{aligned}$$

Proof. (i) From **Lemma 1.2.16** we have that $\{T\leq S\}$ or vice-versa (not sure if I spelled it right) is in $\mathcal{F}_{T\wedge S}$, and from **Lemma 1.2.15** we have $\mathcal{F}_{T\wedge S}\subseteq\mathcal{F}_T$. let $A\subseteq\mathcal{F}_{T\wedge S}$, and consider

$$\int_{A\cap\{T\leq S\}}E[Z|\mathcal{F}_T]d\mathbb{P}=E[Z1_{A\cap\{T\leq S\}}]=\int_{A\cap\{T\leq S\}}E[Z|\mathcal{F}_{S\wedge T}]d\mathbb{P}$$

Now, let $A\in\mathcal{F}_T$ and consider $A\cap\{T\leq S\}$, if we can show it is in $\mathcal{F}_{S\wedge T}$, then we are done. However, we have

$$\begin{aligned}A\cap\{T\leq S\}\cap\{S\wedge T\leq t\} &= A\cap\{T\leq S\}\cap(\{S\leq t\}\cup\{T\leq t\})\\ &= (A\cap\{T\leq S\leq t\})\cup(A\cap\{T\leq S\}\cap\{T\leq t\})\end{aligned}$$

where the second part is in $\mathcal{F}_t$. For the first part we have

$$A\cap\{T\leq S\leq t\}=A\cap\left(\bigcup_{r\in\mathbb{Q}^+\cap[0,t]}\{T\leq r\}\cap\{r\leq S\leq t\}\right)$$

which is in $\mathcal{F}_t$, so we are done.

(ii) $E[Z|\mathcal{F}_{S\wedge T}]=E[E(Z|\mathcal{F}_T)|\mathcal{F}_{T\wedge S}]=E[E(Z|\mathcal{F}_T)|\mathcal{F}_S]$ on $\{S\leq T\}$. On $\{T\leq S\}$ we have $E[E(Z|\mathcal{F}_T)|\mathcal{F}_S]=E[E(Z|\mathcal{F}_{T\wedge S})|\mathcal{F}_S]=E[Z|\mathcal{F}_{T\wedge S}]$. □

Problem (1.2.19). *Let $X_t,\mathcal{F}_t$ be progressively measurable process and T be a stopping time and $f(t,x):[0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ is a bounded $\mathcal{B}([0,\infty))\otimes\mathcal{B}(\mathbb{R}^d)$-measurable function, show that the process $Y_t=\int_0^t f(s,X_s)ds;t\geq 0$ is progressively measurable with repsect to $\mathcal{F}_t$, and Y_T is an $\mathcal{F}_T$ measurable random variable.*

Proof. By **Proposition 1.1.13**, we only need to show Y_t is adapted. Fix $s \geq 0$, then $f(s,x) \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^d)$ measurable, and $\omega \to X_s(\omega) : (\Omega, \mathcal{F}_t) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ measurable, hence the composed function $f(s, X_s(\omega))$ is $\mathcal{F}_s$ measurable. This is true for any $\mathbb{R} \otimes \mathbb{R}^d, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d)$ measurable function. Then by Fubini's theorem, $\omega \to \int_0^t f(s, X_s(\omega))ds$ is also in $\mathcal{F}_t$.

First we have $(s,\omega) \to X_s(\omega) : \{[0,t] \otimes \Omega; \mathcal{B}([0,t]) \otimes \mathcal{F}_t\} \to \mathbb{R}^d \otimes \mathcal{B}(\mathbb{R}^d)$ is measurable. For the second part we need to show for any $B \in \mathcal{B}(\mathbb{R}^d)$, we have $\{Y_T \in B\} \cap \{T \leq t\} \in \mathcal{F}$. By **Proposition 1.2.18**, we know $Y_{T\wedge t}$ is progressively measurable. so

$$\{Y_T \in B\} \cap \{T \leq t\} = \{Y_{T\wedge t} \in B\} \cap \{T \leq t\} \in \mathcal{F}_t$$

□

Problem (1.2.21). *Verify that the class $\mathcal{F}_{T^+}$ is indeed a σ algebra with respect to which T is measurable, that it coincide with $\{A \in \mathcal{F}; A \cap \{T < t\} \in \mathcal{F}_t, \forall t \geq 0\}$, and that if T is a stopping time (so that both $\mathcal{F}_T, \mathcal{F}_{T^+}$ are defined), then $\mathcal{F}_T \subseteq \mathcal{F}_{T^+}$.*

Proof. Recall $\mathcal{F}_{T^+}$ consists of the sets A s.t. $A \cap \{T < t\} \in \mathcal{F}_{t^+}$, since $\mathcal{F}_{t^+}$ itself is a σ-algebra, hence $\bigcap_{n\in\mathbb{N}} A_n \cap \{T < t\} = \bigcap_{n\in\mathbb{N}} (A_n \cap \{T < t\}) \in \mathcal{F}_{t^+}$, and it is obvious that $T \in \mathcal{F}_{T^+}$ for T is a optional.

Now suppose T is a stopping time, and let $A \in \mathcal{F}_T$, then $A \cap \{T < t\} = \mathcal{A} \cap \bigcup_{n=1}\{T \leq t - \frac{1}{n}\} = \bigcup_{n=1} \left(A \cap \{T \leq t - \frac{1}{n}\}\right) \in \mathcal{F}_t \subset \mathcal{F}_{t^+}$ □

Problem (1.2.22). *Verify that analogues of **Lemma 2.15 & 2.16** holds if T and S are assumed to be optional and $\mathcal{F}_T, \mathcal{F}_S$ and $\mathcal{F}_{T\wedge S}$ are replaced by $\mathcal{F}_{T^+}, \mathcal{F}_{S^+}$ and $\mathcal{F}_{(T\wedge S)^+}$, respectively. Prove that if S is an optional time and T is a positive stopping time with $S \leq T$, and $S < T$ on $\{S < \infty\}$, then $\mathcal{F}_{S^+} \subseteq \mathcal{F}_T$.*

Proof. Let $A \in \mathcal{F}_{S^+}$ NTS that $A \cap \{S \leq T\} \in \mathcal{F}_{T^+}$:

$$A \cap \{S \leq T\} \cap \{T < t\} = (A \cap \{S < t\} \cap \{T < t\}) \cap \{T \wedge t \leq S \wedge t\}$$

Now consider the last set

$$\{T \wedge t \leq S \wedge t\} = \bigcup_{t\in\mathbb{Q}^+\cap[0,t]} \{T \wedge t \leq r\} \cap \{r \leq S \wedge t\} \in \mathcal{F}_{t^+}$$

This also means $\mathcal{F}_{T^+} \subset \mathcal{F}_{S^+}$ if $S \geq T$.

Now need to show $\mathcal{F}_{S^+} \cap \mathcal{F}_{T^+} = \mathcal{F}_{(S\wedge T)^+}$: $(\supset)$ relation is obvious from ealier argument. Now let $A \in \mathcal{F}_{S^+} \cap \mathcal{F}_{T^+} \implies A \cap \{S \wedge T < t\} = A \cap \{S < t\} \cap \{T < t\} \in \mathcal{F}_{t^+}$. I will not show the rest since they are pretty standard.

For the last part, let $A \in \mathcal{F}_{S^+}$ so $A \cap \{S < t\} \in \mathcal{F}_{t^+}$.

$$A \cap \{T \leq t\} = \cup_{r\in\mathbb{Q}^+\cap(0,t)} A \cap \{S < r < T \leq t\}$$

□

Problem (1.2.23). *Show that if $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of optional times and $T = \inf_{n\geq 1} T_n$, then $\mathcal{F}_{T^+} = \bigcap_{n=1}^{\infty} \mathcal{F}_{T_n^+}$. Besides, if each T_n is a positive stopping time and $T < T_n$ on $\{T < \infty\}$, then we have $\mathcal{F}_{T^+} = \cap_{n=1}^{\infty} \mathcal{F}_{T_n}$.*

Proof. By assumption, T is of course an optional time, since we have $T \leq T_n$, we also have $\mathcal{F}_{T^+} \subseteq \mathcal{F}_{T_n^+}$ for all $n \geq 1$, hence it is also a subset of the intersections. Now suppose $A \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{T_n^+}$, so

$$A \cap \{T_n < t\} \in \mathcal{F}_{t^+} \quad \forall n \in \mathbb{N}.$$

Conisder

$$\begin{aligned} A \cap \{T < t\} &= A \cap \bigcup_{n\in\mathbb{N}} \{T_n < t\} \\ &= \bigcup_{n\in\mathbb{N}} A \cap \{T_n < t\} \in \mathcal{F}_{t^+} \end{aligned}$$

Now suppose T_n's are positive stopping times, then by **Problem 1.2.22**, we have $\mathcal{F}_{T^+} \subset \bigcap_{n\in\mathbb{N}} \mathcal{F}_{T_n}$. Let $A \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{T_n}$, we have

$$A \cap \{T < t\} = A \cap \bigcup_{n\in\mathbb{N}} \{T_n < t\} \in \mathcal{F}_t$$

□

Problem (1.2.24). *Given an optional time T of the filtration $\{\mathcal{F}_t\}$, consider the sequence $\{T_n\}_{n\in\mathbb{N}}$ of random times given by*

$$T_n(\omega)\begin{cases} T(\omega); & on\ \{\omega : T(\omega)=\infty\} \\ \frac{k}{2^n}; & on\ \{\omega : \frac{k-1}{2^n} \le T(\omega) < \frac{k}{2^n}\} \end{cases}$$

for $n \ge 1$, $k \ge 1$. Obviously $T_n \ge T_{n+1} \ge T$ for every $n \ge 1$. Show that each T_n is a stopping time, that $\lim_{n\to\infty} T_n = T$, and that for every $A \in \mathcal{F}_{T^+}$ we ahve $A \cap \{T_n = \frac{k}{2^n}\} \in \mathcal{F}_{\frac{k}{n^2 2}}, n, k \ge 1$.

Proof.

$$\{T_n \le t\} = \{T < \max_{k\in\mathbb{N}}\{\frac{k}{2^n} : \frac{k+1}{2^n} < t\}\} \in \mathcal{F}_t$$

hence they are all stopping times. For each $\omega \in \Omega$, for all $\epsilon > 0$, there exists $n \ge 1$ such that $\frac{1}{2^n} < \epsilon$ with $T_n(\omega) - T(\omega) = \frac{k}{2^n} - T(\omega) \le \frac{1}{2^n}$ for some $k \ge 1$, hence we have the convergence. Now suppose $A \in \mathcal{F}_{T^+}$, consider

$$A \cap \{T_n = \frac{k}{n^2}\} = A \cap \{\frac{k-1}{n^2} \le T < \frac{k}{2^n}\} \in \mathcal{F}_{\frac{k}{n^2}}.$$

□

1.3 Continuous-Time Martingales

Problem (1.3.2). *Let $T_1, T_2, ...$ be a sequence of independent, exponential distributed random variables with parameter $\lambda > 0$:*

$$\mathbb{P}[T_i \in dt] = \lambda e^{-\lambda t} dt, \quad t \ge 0$$

Let $S_0 = 0$ and $S_n = \sum_{i=1}^n T_i; n \ge 1$ (We may think of S_n as the time at which the n-th customer arrives in a queue, and of the random variable $T_i, i \in \mathbb{N}$ as the interarrival times.) Define a continuous-time, integer valued RCLL process

$$N_t = \max\{n \ge 0; S_n \le t\}; \quad 0 \le t < \infty$$

(We may regard N_t as the number of customers who arrive up to time t.)

(i) Show that for $0 \le s < t$ we have

$$\mathbb{P}[S_{N_s+1} > t | \mathcal{F}_s^N] = \exp(-\lambda(t-s)), \quad a.s.\mathbb{P}$$

(Hint: Choose $\widetilde{A} \in \mathcal{F}_s^N$ and a nonegative integer n. Show that there exists an event$A \in \sigma(T_1, ..., T_n)$ such that $A \cap \{N_s = n\} = \widetilde{A} \cap \{N_s = n\}$, and use independence between T_{n+1} and the pair $(S_n, 1)$ to establish

$$\int_{\widetilde{A}\cap\{N_s=n\}} \mathbb{P}[S_{n+1} > t|\mathcal{F}_s^N] d\mathbb{P} = \exp(-\lambda(t-s))\mathbb{P}[\widetilde{A} \cap \{N_s = n\}])$$

(ii) Show that for $0 \le s < t$, $N_t - N_s$ is a Poisson random variable with parameter $\lambda(t-s)$, independent of $\mathcal{F}_s^N$. (Hint: with $\widetilde{A} \in \mathcal{F}_s^N$ and $n \ge 0$ as before, use the result in (i) to estabilish

$$\int_{\widetilde{A}\cap\{N_s=n\}} \mathbb{P}[N_t - N_s \le k|\mathcal{F}_s^N] d\mathbb{P} = \mathbb{P}[\widetilde{A} \cap \{N_s = n\}] \cdot \sum_{j=0}^k \exp(-\lambda(t-s)) \frac{(\lambda(t-s))^j}{j!}$$

for every integer $k \ge 0$.)

Proof. (i) Use the hint and the solution, consider $T_N(s) = "\mathcal{F}_s^N | \{N_s = n\}"$, recall that $\mathcal{F}_s^N = \sigma(N_t; 0 \le t \le s)$, so $T_N(s)$ is generated by the family of sets of the form

$$\{N_{s_1} < n_1, ..., N_{s_k} < n_k, N_s = n\}; \quad 0 \le s_1 \le ... \le s_k \le s$$

Similarly, consider $S_T(s) = \sigma(T_1, ..., T_n) | \{N = n\}$ which is generated by the sets of the form $\{T_1 \le t_1, ..., T_n \le t_n, N_s = n\}$ where $\sum_{k=1}^n t_k \le s$, or

$$\{S_1 \le t_1, ..., S_n \le t_n, N_s = n\} \quad 0 \le t_1 \le ... \le t_n \le s.$$

Note that $\{N_t \geq k\} = \{S_k \leq t\}$, hence $T_N(s) = S_T(s)$ since they have the same generating sets. Therefore, for all $\widetilde{A} \in \mathcal{F}_s^N \cap \{N_s = n\}$, there exists $A \in \sigma(T_1, .., T_n) \cap \{N_s = n\}$ such that $\widetilde{A} = A$. Now conisder

$$\int_{\widetilde{A}\cap\{N_s=n\}} \mathbb{P}[S_{n+1} > t|\mathcal{F}_s^N]d\mathbb{P} = \mathbb{P}[S_{n+1} > t \cap A \cap \{N_s = n\}]$$

Proof needs to be filled

$$\begin{aligned}
&= \mathbb{P}[S_n + T_{n+1} > t \cap A \cap S_n \leq s < S_{n+1}] \\
&= \int_{t-s}^{\infty} \mathbb{P}[S_n > t-u, A, S_n \leq s]\lambda e^{-\lambda u} du \\
&= e^{-\lambda(t-s)} \int_0^{\infty} \mathbb{P}[S_n > s-u, A, S_n \leq s]\lambda e^{-\lambda u} du \\
&= e^{-\lambda(t-s)}\mathbb{P}[S_n + T_{n+1} > s \geq S_n, A] \\
&= e^{-\lambda(t-s)}\mathbb{P}[N_s = n \cap \widetilde{A}].
\end{aligned}$$

Now sum over all n's to get the desired answer.

(ii) Use the hint:

$$\begin{aligned}
\int_{\widetilde{A}\cap\{N_s=n\}} \mathbb{P}[N_t - N_s \leq k|\mathcal{F}_s^N]d\mathbb{P} &= \mathbb{P}[A \cap \{N_s = n\} \cap N_t - N_s \leq k] \\
&= \mathbb{P}[A \cap \{N_s = n\} \cap N_t < k+n+1] \\
&= \mathbb{P}[A \cap \{N_s = n\} \cap S_{k+n+1} > t] \\
&= \mathbb{P}[A \cap \{N_s = n\} \cap S_{n+1} + \sum_{j=n+2}^{n+k+1} T_j > t] \quad \text{let } Z \text{ be the summation} \\
&= \mathbb{P}[A \cap \{N_s = n\} \cap S_{n+1} > t - Z] \\
&= \int_0^{\infty} \mathbb{P}[A \cap \{N_s = n\} \cap S_{n+1} > t-u]\mathbb{P}[Z \in du] \\
&= \int_0^{\infty} \mathbb{P}[A \cap S_n \leq s, S_{n+1} > s, S_{n+1} > t-u]\mathbb{P}[Z \in du] \\
&= \int_0^{t-s} \mathbb{P}[A \cap S_n \leq s, S_{n+1} > t-u]\mathbb{P}[Z \in du] + \mathbb{P}[A \cap S_n \leq s, S_{n+1} > s, Z \geq t-s] \\
&= \int_0^{t-s} \mathbb{P}[A, S_n \leq s, S_{n+1} > t-u]d\mathbb{P}[Z \in du] + \mathbb{P}[A, \{N_s = n\}]\mathbb{P}[Z \geq t-s]
\end{aligned}$$

Note that sume of exponential r.v.'s has gamma distributions, that is $\mathbb{P}[Z \in du] = \frac{[\lambda u]^{k-1}}{(k-1)!}\lambda e^{-\lambda u}$ and

$$\mathbb{P}[Z > \theta] = \sum_{j=0}^{k-1} \frac{(\lambda\theta)^j}{j!} e^{-\lambda\theta}$$

So the above is equal to

$$\begin{aligned}
&\int_0^{t-s} \mathbb{P}[A \cap \{N_s = n\}, S_{n+1} \geq t-u]d\mathbb{P}[Z \in du] + \mathbb{P}[A, \{N_s = n\}]\mathbb{P}[Z \geq t-s] \\
&= \int_0^{t-s} \mathbb{P}[A \cap \{N_s = n\}]\mathbb{P}[T_{n+1} > t-s-u]\mathbb{P}[Z \in du] + \mathbb{P}[A, \{N_s = n\}]\mathbb{P}[Z \geq t-s]
\end{aligned}$$

Then put everything together, we have the desired result. still a bit unsure about the last equality □

Problem (1.3.4). *Prove that a compensated Poisson process $\{M_t, \mathcal{F}_t; t \geq 0\}$ is a martingale.*

Proof. let $t \geq s$ and consider

$$\begin{aligned}
\mathbb{E}[M_t - M_s|\mathcal{F}_s] &= \mathbb{E}[N_t - N_s + \lambda(t-s)|\mathcal{F}_s] \\
&= \mathbb{E}[N_t - N_s] - \lambda(t-s) = 0
\end{aligned}$$

□

Problem (1.3.7). *let* $\{X_t = (X_t^{(1)}, ..., X_t^{(d)}) \in \mathbb{R}^n, \mathcal{F}_t; 0 \le t < \infty\}$ *be a vector of martingales, and* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *a convex function with* $\mathbb{E}[|\varphi(X_t)|] < \infty$ *for all* $t \ge 0$*. Show that* $\{\varphi(X_t) : \mathcal{F}_t; 0 \le t < \infty\}$ *is a submartingale; In particular,* $\{\|X_t\|; \mathcal{F}_t; 0 \le t < \infty\}$ *is a submartingale.*

Proof. Using Jessen's Inequality, let $s \le t$, we have

$$\varphi(X_s) = \varphi(\mathbb{E}[X_t|\mathcal{F}_s]) \le \mathbb{E}[\varphi(X_t)|\mathcal{F}_s]$$

Proof from solution: Due to convexity, there exists a family $\{h_\alpha\}$ such that $\varphi = \sup_\alpha h_\alpha$, where h_α are linear functions sends $\mathbb{R}^n \to \mathbb{R}$. Therefore,

$$\mathbb{E}[\varphi(X_t)|\mathcal{F}_s] \ge \mathbb{E}[h_\alpha(X_t)|\mathcal{F}_s] = h_\alpha(X_s) \quad \forall \alpha$$

hence $\mathbb{E}[\varphi(X_t)|\mathcal{F}_s] \ge \varphi(X_s)$. ☐

Problem (1.3.9). *Let* N *be a Poisson process with intensity* λ*.*

(a) For any $c > 0$,

$$\overline{\lim_{t\to\infty}} \mathbb{P}[\sup_{0\le s\le t} (N_s - \lambda s) \ge c\sqrt{\lambda t}] \le \frac{1}{c\sqrt{2\pi}}$$

(b) For any $c > 0$,

$$\overline{\lim_{t\to\infty}} \mathbb{P}\left[\inf_{0\le s\le t} (N_s - \lambda s) \le -c\sqrt{\lambda t}\right] \le \frac{1}{c\sqrt{2\pi}}$$

(c) for $0 < \sigma < \tau$*, we have*

$$\mathbb{E}\left[\sup_{\sigma\le t\le\tau} \left(\frac{N_t}{t} - \lambda\right)^2\right] \le \frac{4\tau\lambda}{\sigma^2}$$

Hint: Use Stirling's Approximation to show that $\lim_{t\to\infty} \frac{1}{\sqrt{\lambda t}} \mathbb{E}[N_t - \lambda t]^+ = \frac{1}{\sqrt{2\pi}}$

Proof. $N_t - \lambda t$, the compensated Poisson process is a Martingale, hence by the first submartingale inequality, we have

$$\mathbb{P}[\sup_{0\le s\le t} (N_s - \lambda s) \ge c\sqrt{\lambda t}] \le \frac{\mathbb{E}(N_t - t\lambda)^+}{c\sqrt{\lambda t}}$$

For large t we have

$$\begin{aligned}
\frac{1}{\sqrt{\lambda t}} \mathbb{E}[N_t - \lambda t]^+ &= \sqrt{t} \sum_{n=\lceil \lambda t \rceil}^{\infty} \frac{(t\lambda)^n}{n!} e^{-t\lambda} (\frac{n}{t} - \lambda) \frac{1}{\sqrt{\lambda}} \\
&= \sqrt{t}\sqrt{\lambda} e^{-\lambda t} \frac{\lfloor t\lambda \rfloor^{\lfloor t\lambda \rfloor}}{(\lfloor t\lambda \rfloor)!} \\
&\approx \sqrt{t}\sqrt{\lambda} e^{-\lambda t} \frac{(t\lambda)^{t\lambda}}{(t\lambda)!} \\
&\to \frac{1}{\sqrt{2\pi}} \quad \text{as } t \to \infty
\end{aligned}$$

Therefore, take the limit as $t \to \infty$ on both side of the above inequality, we have the desired result.

(b) Using the second submartingale inequality and the stirling approximation result above, the proof is bassically the same.

(c)

$$\mathbb{E}\left[\sup_{\sigma\le t\le\tau} \left(\frac{N_t}{t} - \lambda\right)^2\right] \le \mathbb{E}\left[\sup_{\sigma\le t\le\tau} (N_t - \lambda t)^2\right] \frac{1}{\sigma^2}$$

and by Jessen's $(N_t - \lambda t)$ is a submartingale, hence by Doob's Maximum Inequality we have

$$\mathbb{E}\left[\sup_{\sigma\le t\le\tau} (N_t - \lambda t)^2\right] \le 4\mathbb{E}[X_\tau^2] = 4\tau\lambda$$

The above two inequalities gives the desired result. ☐

Problem (1.3.11). *Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a decreasing sequence of sub-σ-fields of $\mathcal{F}$ (i.e., $\mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \mathcal{F}, \forall n \geq 1$), and let $\{X_n, \mathcal{F}_n, 1 \leq n < \infty\}$ be a **Backward Submartingale**; i.e., $\mathbb{E}[|X_n|] < \infty$ for all nand $X_n \in \mathcal{F}_n$ with $\mathbb{E}[X_n|\mathcal{F}_{n+1}] \geq X_{n+1}$ a.s. $\mathbb{P}$ for all n. Show $l \triangleq \lim_{n\to\infty} \mathbb{E}[X_n] > -\infty$ implies that the sequence $\{X_n\}$ is uniformly integrable.*

Proof. Note that $^+ : x \to \max\{x, 0\}$ is a convex function, then by Jessen's Inequality, X_n^+ is also a backward submartingale. Since $\mathbb{E}[|X_n^+|] \leq \mathbb{E}[X_1^+]$ for all n. By Markov's Inequality we have $\lambda \mathbb{P}[|X_n| > \lambda] = \mathbb{E}[|X_n|] = \mathbb{E}[X_n^+] + \mathbb{E}[X_n^-] = 2\mathbb{E}[X_n^+] - \mathbb{E}[X_n] \leq 2\mathbb{E}[X_1^+] - l < \infty$, so $\mathbb{P}[|X_n| > \lambda] \to 0$ as λ goes to ∞. So consider

$$\mathbb{E}[X_n^+ 1_{X_n^+ > \lambda}] \leq \mathbb{E}[X_1^+ 1_{X_n^+ > \lambda}] \leq \mathbb{E}[X_1^+ 1_{|X_n| > \lambda}]$$

which goes to zero as $\lambda \to \infty$.

Now let $m < n$ and consider

$$\begin{aligned} 0 \geq \int_{\{X_n < -\lambda\}} X_n &= \mathbb{E}[X_n] - \int_{\{X_n \geq -\lambda\}} X_n \\ &\geq \mathbb{E}[X_n] - \int_{\{X_n \geq -\lambda\}} X_m \\ &= \mathbb{E}[X_n] - \mathbb{E}[X_m] + \int_{\{X_m < -\lambda\}} X_m \end{aligned}$$

since $\lim_{n\to\infty} \mathbb{E}[X_n]$ converges, so we can take m large so that $\int_{\{|X_m| > \lambda\}} X_m^- < \epsilon$, also, for all $n \geq m$, we have

$$-\epsilon \leq \int_{\{X_m \leq -\lambda\}} X_m \leq \int_{\{X_n \leq -\lambda\}} X_n \leq 0$$

so the negative part is also uniformly integrable. □

Problem (1.3.16). *Let $\{X_t; \mathcal{F}_t, 0 \leq t < \infty\}$ be a right-continuous, nonnegative supermartingale; show $X(\omega) = \lim_{t\to\infty} X_t(\omega)$ exists for $\mathbb{P}$-a.s. $\omega \in \Omega$, and $X_t, \mathcal{F}_t, 0 \leq t \leq \infty$ forms a supermartingale.*

Proof. Note that $-X_t$ is a right-continuous submartingale with $\sup_t \mathbb{E}[X_t^+] = 0$. Hence by Submartingale convergence, we are done. □

Excercise (1.3.18). *Suppose $\mathcal{F}_t$ satisfies the usual conditions. Then every right-continuous, uniformly integrable supermartingales $\{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ admits the Riesz decomposition $X_t = M_t + Z_t$, a.s. $\mathbb{P}$, as the sume of right continuous, uniformly integrable martingale M and a potential Z.*

Proof. By uniformly integrability and Mtg convergence theorem, there is a last element call X_∞ such that $\mathbb{E}[X_t] \geq \mathbb{E}[X_\infty|\mathcal{F}_t]$. Define $A_t \triangleq X_t - \mathbb{E}[X_\infty|\mathcal{F}_t]$ so A_t is a supermartingale that converges to zero a.s. and in L^1 (monotone), hence it is a potential, and $M_t \triangleq \mathbb{E}[X_\infty|\mathcal{F}_t]$ is obviously a Mtg. Where right-continuous used in Mtg convergence theorem, and usual condition used in defining X_∞ since pointwise convergence fails only in a null set. □

Problem (1.3.19). *The following three conditions are equivalent for nonnegative, right-continuous submartingale $\{X_t; \mathcal{F}_t; 0 \leq t < \infty\}$:*

1. *it is uniformly integrable family of random variables;*
2. *it converges in L^1, as $t \to \infty$;*
3. *it converges $\mathbb{P}$ a.s. (as $t \to \infty$) to an integrable random variable X_∞, such that $\{X_t; \mathcal{F}_t, 0 \leq t \leq \infty\}$ is a submartingale.*

Proof. (1) ⇒ (2): Uniformly integrability ⇒ $\exists M \geq 0; \sup \mathbb{E}[|X_t|] \leq M$. Hence by Mtg convergence, we have almost sure convergence, call the convergent element X_∞. Fatou's lemma to get $\lim \mathbb{E}[X_t] = \liminf_{t\to\infty} \mathbb{E}[X_t] \geq \mathbb{E}[\liminf_{t\to\infty} X_t] = \mathbb{E}[\lim_{t\to\infty} X_t] = \mathbb{E}[X_\infty]$. However, $\mathbb{E}[X_t]$ is increasing in t, so we have L^1 convergence.

(2) ⇒ (3): Convergence in L^1 implies convergence in probability. Now, let $s \geq 0$ and $A \in \mathcal{F}_s$, then $\int_A X_\infty d\mathbb{P} = \lim_{t\to\infty} \int_A X_t d\mathbb{P} \leq \int_A X_s d\mathbb{P}$. Hence it is sub Mtg with last element.

(3) ⇒ (1): $Y \triangleq \mathbb{E}[X_\infty|\mathcal{F}_t] \geq X_t$, and since positive, $\mathbb{E}[X_t 1_{\{X_t > \lambda\}}] \leq \mathbb{E}[Y 1_{\{X_t > \lambda\}}]$. Now note that $\lambda \mathbb{P}[X_t > \lambda] \leq \mathbb{E}[X_t] \leq \mathbb{E}[X_\infty]$ so $\sup_t \mathbb{P}[X_t > \lambda] \to 0$ as $\lambda \to 0$. □

Problem (1.3.20). *The following four conditions are equivalent for a right-continuous martingale $\{X_t; \mathcal{F}_t; 0 \leq t < \infty\}$:*

- *(1),(2) as previous problem.*
- *(3) it converges $\mathbb{P}$ a.s. (as $t \to \infty$) to an integrable random variable X_∞, such that $\{X_t; \mathcal{F}_t; 0 \leq t \leq \infty\}$ is a martingale.*

- *(4) there exists a integrable random variable Y such that $X_t = \mathbb{E}[Y|\mathcal{F}_t]$ a.s. $\mathbb{P}$, for all $t \geq 0$.*

Besides, if (4) holds and X_∞ is the random variable in (3), then

$$\mathbb{E}[Y|\mathcal{F}_\infty] = X_\infty \quad a.s.\ \mathbb{P}$$

Proof. Note that (1) to (2) to (3) is shown in the previous problem since we did not use positivity for those implications, also, " $\leq$ " case is due to the similar properties of super Mtg's. For (3) to (4), set $X_\infty = Y$. For (4) to (1), $|\cdot|\mathbb{R} \to \mathbb{R}^+$ is convex function, then by **Proposition 3.6** or just Jessen's Inequality, $|\mathbb{E}[Y|\mathcal{F}_t]|$ is a sub Mtg and expectation achieves maximum at $t = 0$, hence it is uniformly integrable.

The beside part is easily seen. □

Problem (1.3.21). *Let $\{N_t; \mathcal{F}_t; 0 \leq t < \infty\}$ be a Poisson process with parameter $\lambda > 0$. For $u \in \mathbb{C}$ and $i = \sqrt{1}$, define the process*

$$X_t = \exp\left\{iuN_t - \lambda t(e^{iu} - 1)\right\}$$

(i) Show that $\Re(X_t)$ and $\Im(X_t)$ are martingales.
(ii) Consider X with $u = -i$. Does this martingale satisfy the equivalent conditions of problem 3.20?

Proof. (i) Let $s \leq t$, and note we don't have to check real and imaginary parts seperately,

$$\begin{aligned}\mathbb{E}[X_t - X_s|\mathcal{F}_s] &= \mathbb{E}[X_s\left(\exp\left\{iu(N_t - N_s) - \lambda(t-s)(e^{iu} - 1)\right\} - 1\right)|\mathcal{F}_s] \\ &= X_s\mathbb{E}[\left(\exp\left\{iu(N_t - N_s) - \lambda(t-s)(e^{iu} - 1)\right\} - 1\right)]\end{aligned}$$

where the last equality is by independence. $N_t - N_s$ is a Poisson random variable with parameter $\lambda(t-s)$. By looking up the character function of Poisson random variable, we found the expectation is zero, hence a martingale.

(ii) Let $u = -i$, then

$$X_t = \exp\left\{N_t - \lambda t(e-1)\right\}$$

and it is distributed as follows

$$\mathbb{P}[X_t = \exp\left\{n - \lambda t(e-1)\right\}] = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

So, if we were to take $1_{\{X_t > K\}}$, it is enough to consider the expectation on $\{N_t > K\}$, so consider

$$\mathbb{E}[X_t 1_{N_t \geq K}] = \sum_{n=k}^{\infty} \frac{(e\lambda t)^n}{n!} e^{-\lambda te}$$

each summant is less than 1, so by bounded convergence theorem, take $k \to \infty$ it converges to zero, so it does satisfy the equivalent conditions of the previous problem. □

1.3.C Optional Sample Theorem

Problem (1.3.32). *Establish the optional sampling theorem fro a right-continuous submartingale $\{X_t; \mathcal{F}_t; 0 \leq t < \infty\}$ and optional times $S \leq T$ under either of the following two conditions:*

- *T is a bounded stopping time (there exists a number $a > 0$ such that $T \leq a$);*
- *there exists an integrable random variable Y, such that $X_t \leq \mathbb{E}[Y|\mathcal{F}_t]$ a.s. $\mathbb{P}$, for every $t \geq 0$.*

Proof. (i) Let $a \geq T(\omega)$ for all ω, then $Y_t = X_{t\wedge a}, \mathcal{F}_t, 0 \leq t < \infty$ is a submartingale with last element X_a. So by **Theorem 1.3.22** we have $\mathbb{E}[X_T|\mathcal{F}_{S^+}] = \mathbb{E}[Y_T|\mathcal{F}_{S^+}] \geq Y_S = X_S$.
(ii) This is the definition of sub Mtg with last element, so use **Theorem 1.3.22** directly. □

Problem (1.3.24). *Suppose thath $\{X_t; \mathcal{F}_t; 0 \leq t < \infty\}$ is a right-continuous sub Mtg and $S \leq T$ are stopping times of $\mathcal{F}_t$. Then*

- *$\{X_{T\wedge t}; \mathcal{F}_t; 0 \leq t < \infty\}$ is a sub-Mtg;*
- *$\mathbb{E}[X_{T\wedge t}|\mathcal{F}_S] \geq X_{S\wedge t}$ a.s. $\mathbb{P}$, for every $t \geq 0$.*

Proof. (i) Let $s \le t$, and $t \wedge T$ and $s \wedge T$ are bounded stopping times. Optional stopping theorem tells us $\mathbb{E}[X_{T\wedge t}|\mathcal{F}_{T\wedge s}] \ge X_{T\wedge s}$.

$$\begin{aligned}\mathbb{E}[X_{T\wedge t}|\mathcal{F}_s] &= \mathbb{E}[1_{\{T\le s\}}X_{T\wedge t}|\mathcal{F}_s] + \mathbb{E}[1_{\{T>s\}}X_{T\wedge t}|\mathcal{F}_s] \\ &= \mathbb{E}[1_{\{T\le s\}}X_{T\wedge s}|\mathcal{F}_s] + \mathbb{E}[1_{\{T>s\}}X_{T\wedge t}|\mathcal{F}_s] \\ &\ge \mathbb{E}[1_{\{T\le s\}}X_{T\wedge s}|\mathcal{F}_s] + 1_{\{T>s\}}X_s \\ &= 1_{\{T\le s\}}X_T + 1_{\{T>s\}}X_s \\ &= X_{T\wedge s}\end{aligned}$$

To justify the third inequality, let $A \in \mathcal{F}_s$, then $A \cap \{T > s\} \in \mathcal{F}_s \cap \mathcal{F}_T$

$$\mathbb{E}[1_{A\cap\{T>s\}}X_{T\wedge t}] = \mathbb{E}[\mathbb{E}[1_{A\cap\{S>t\}}X_{T\wedge t}|\mathcal{F}_{s\wedge T}]] \ge \mathbb{E}[1_{A\cap\{T>s\}}X_{T\wedge s}]$$

The second equality is because $X_{T\wedge s} \in \mathcal{F}_{T\wedge s} \subset \mathcal{F}_s$ by **Proposition 1.2.18**.
(ii) Note $\{T \le s\} \in \mathcal{F}_s$.

$$\begin{aligned}\mathbb{E}[X_{T\wedge t}|\mathcal{F}_S] &= \mathbb{E}[1_{S\le t}X_{T\wedge t}|\mathcal{F}_S] + \mathbb{E}[1_{S>t}X_{T\wedge t}|\mathcal{F}_S] \\ &= \mathbb{E}[1_{\{S\le t\}}X_{T\wedge t}|\mathcal{F}_S] + \mathbb{E}[1_{\{S>t\}}X_{S\wedge t}|\mathcal{F}_S]\end{aligned}$$

note $\mathbb{E}[1_{\{S>t\}}X_{S\wedge t}|\mathcal{F}_S] = 1_{\{S>t\}}\mathbb{E}[X_S|\mathcal{F}_S] = 1_{\{S>t\}}X_S$. And $\mathbb{E}[1_{\{S\le t\}}X_{T\wedge t}|\mathcal{F}_S] \ge 1_{\{S\le t\}}X_{S\wedge t}$: to justify this, let $A \in \mathcal{F}_S$, then $A \cap \{S \le t\} \in \mathcal{F}_{S\wedge t}$ since $A \cap \{S \le t\} \in \mathcal{F}_t$ and $A, \{S \le t\} \in \mathcal{F}_S$ and $\mathcal{F}_S \cap \mathcal{F}_t = \mathcal{F}_{S\wedge t}$, and consider the following integral

$$\int_A 1_{\{S\le t\}}X_{T\wedge t}d\mathbb{P} = \mathbb{E}[\mathbb{E}[1_{A\cap\{S\le t\}}X_{T\wedge t}|\mathcal{F}_{S\wedge t}]] \ge \mathbb{E}[1_{A\cap\{S\le t\}}X_{S\wedge t}]$$

True for all $A \in \mathcal{F}_S$, hence we have desired inequality. □

Problem (1.3.25)**.** *A sub Mtg of constant expectation for all $t \ge 0$ is a Mtg.*

Proof.

$$\mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_s] - X_s] = 0$$

and $\mathbb{E}[X_t|\mathcal{F}_s] \ge X_s$, so we must have equality almost everywhere. □

Problem (1.3.26)**.** *A right-continous process $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ with $\mathbb{E}[|X_t|] < \infty$ for all $t \ge 0$ is a submartingale if and only if for every pair $S \le T$ of bounded stopping times of the filtration $\mathcal{F}_t$ we have*

$$\mathbb{E}[X_T] \ge \mathbb{E}[X_S]$$

Proof. Why isn't this obvious? □

Problem (1.3.27)**.** *Let T be a bounded stopping time of the filtration $\mathcal{F}_t$, which satisfies the usual conditions, and define $\widetilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$. Then $\widetilde{\mathcal{F}}_t$ also satifies the usual conditions.*
(i) If $X = \{X_t; \mathcal{F}_t; 0 \le t < \infty\}$ is right-continuous submartingale, then so is $\widetilde{X} = \{\widetilde{X}_t \triangleq X_{T+t} - X_T, \widetilde{\mathcal{F}}_t, 0 \le t < \infty\}$.
(ii) If $\widetilde{X} = \{\widetilde{X}_t, \widetilde{\mathcal{F}}_t, 0 \le t < \infty\}$ is a right continous submartingale with $\widetilde{X}_0 = 0$ a.s. $\mathbb{P}$, then $X = \{\widetilde{X}_{t-T\vee 0}; \mathcal{F}_t; 0 \le t < \infty\}$ is also a submartingale.

Proof. (i) There is no doubt about adaptivity. Let $s \le t$ and consider

$$\mathbb{E}[X_{T+t} - X_T|\mathcal{F}_{T+s}] = \mathbb{E}[X_{T+t}|\mathcal{F}_{T+s}] - X_T \ge X_{T+s} - X_T$$

by Optional Sampling.

(ii) Problem with this question: $\mathcal{F}_{t-T}$ not formally defined in the previous text.

$X_t = \widetilde{X}_{(t-T)\vee 0}$ a.s. $\mathbb{P}$. For adaptivity, $A \in \widetilde{\mathcal{F}}_t = \mathcal{F}_{T+t} \iff A \cap \{T + t \le s\} \in \mathcal{F}_s$, and we know $\{\widetilde{X}_s \in A\} \in \widetilde{\mathcal{F}}_t = \mathcal{F}_{t+T}$ for any $A \in \mathcal{B}(\mathbb{R}^n)$, that is $\{\widetilde{X}_t \in A\} \cap \{T + t \le s\} \in \mathcal{F}_s$ for all $s \ge 0$.

$$X_t = \widetilde{X}_{(t-T)\vee 0} = \widetilde{X}_{(t-T)\vee 0}(1_{t<T} + 1_{t\ge T}) = \widetilde{X}_0 1_{t<T} + \widetilde{X}_{t-T}1_{t\ge T}$$

That means if $A \in \mathcal{B}(\mathbb{R}^n)$, then

$$\begin{aligned}\{X_t \in A\} &= \{\widetilde{X}_0 1_{t<T} + \widetilde{X}_{t-T} 1_{t\geq T} \in A\} \\ &= \{\widetilde{X}_0 1_{t<T} \in A\} \cup \{\widetilde{X}_{t-T} 1_{t\geq T} \in A\}\end{aligned}$$

where the first term is cerntainly in $\mathcal{F}_t$. $\{1_{t\geq T}\} \in \mathcal{F}_t$ □

Problem (1.3.28). *Let $Z = \{Z_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a continuous nonnegative martingale with $Z_\infty = \lim_{t\to\infty} Z_t = 0$ a.s. $\mathbb{P}$. Show that for every $s \geq 0, b > 0$ we have*

1. $\mathbb{P}\left[\sup_{t>s} Z_t \geq b | \mathcal{F}_s\right] = \frac{1}{b} Z_s$, *a.s. on* $\{Z_s < b\}$.
2. $\mathbb{P}\left[\sup_{t\geq s} Z_t \geq b\right] = \mathbb{P}[Z_s \geq b] + \frac{1}{b}\mathbb{E}[Z_s 1_{Z_s<b}]$

Proof. (1) Let $B = \{Z_s < b\}$ and let $T \triangleq \inf_t\{t \geq s : Z_t = b\}$, then $Z_{t\wedge T}$ is a martingale, and by one version of optional sampling, for all $A \in \mathcal{F}_s$ we have

$$\begin{aligned}\int_{A\cap B} Z_s d\mathbb{P} &= \int_{A\cap B} Z_{t\wedge T} d\mathbb{P} \\ &= \int_{A\cap B} 1_{t\geq T} b d\mathbb{P} + \int_{A\cap B} 1_{t<T} Z_t d\mathbb{P}\end{aligned}$$

Now note the second term is monotone in t, so send $t \to \infty$ to get

$$\int_{A\cap B} Z_s d\mathbb{P} = b\mathbb{P}[A \cap \{Z_s < b\} \cap \{T < \infty\}]$$

since this is true for all $A \in \mathcal{F}_s$, then we are done.

(2) follows directly from (1). □

Problem (1.3.29). *let $\{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a continuous, nonnegative super martingale and $T = \inf\{t \geq 0; X_t = 0\}$. Show that*

$$X_{T+t} = 0; \quad 0 \leq t < \infty \quad \textit{holds a.s. on } \{T < \infty\}$$

Proof. Note that $\{X_t\}$ is uniformly integrable, hence it has a last element, also note that optional sampling theorem applies to $-X_t$ since it is a submartingale.

$$0 \geq -\mathbb{E}[1_{T<\infty} X_{T+t}] = -\mathbb{E}[\mathbb{E}[X_{T+t}|\mathcal{F}_T] 1_{T<\infty}] \geq -\mathbb{E}[X_T 1_{T<\infty}] = 0$$

and the result is given by positivity. □

Problem (1.5.7). *Show $< \cdot, \cdot >$ is a bilinear form of $\mathcal{M}_2$, i.e. for any members $X, Y, Z \in \mathcal{M}_2$ and real number α, β, we have*

1. $<\alpha X + \beta Y, Z> = \alpha < X, Z > + \beta < Y, Z >$,
2. $< X, Y > = < Y, X >$
3. $| < X, Y > |^2 \leq < X > < Y >$
4. *For P-a.e.* $\omega \in \Omega$

$$\widehat{\xi}_t(\omega) - \widehat{\xi}_s(\omega) \leq \frac{1}{2}[< X >_t (\omega) - < X >_s (\omega) + < Y >_t (\omega) - < Y >_s (\omega)] \quad 0 \leq s < t < \infty$$

where $\widehat{\xi}_t$ denote the total variation of $\xi \triangleq < X, Y >$ on $[0, t]$.

Proof. (1) $\alpha XZ + \beta YZ - < \alpha X + \beta Y, Z >$ is a martingale, and $\alpha XZ + \beta YZ - \alpha < X, Z > -\beta < Y, Z >$ is also an martinagle, and by the uniqueness of the cross variation, they are equal.

(2) Since multiplication in $\mathcal{M}^2$ is commutative.

(3)

$$0 \leq < X - \lambda Y > = < X + \lambda Y, X - \lambda Y >$$
$$= < X > + \lambda^2 < Y > - 2\lambda < X, Y >$$

Assume WLOG that $< Y >_t > 0$ for $t > 0$, since if $< Y >_t$ is zero, for $t \in [0, T]$ for some T, Then $\mathbb{E}[Y^2] = \mathbb{E}[< Y >_t] = 0$, since $Y^2 \geq 0$, then Y is identically zero on the interval $[0, T]$, hence both quadratic and cross variations are zeros, so equality holds. With that cleared, let $\lambda = \frac{<X,Y>}{<Y>}$, then the above is

$$0 \leq < X > + \frac{< X, Y >^2}{< Y >} - 2\frac{< X, Y >^2}{< Y >}$$

Multiply both sides by $< Y >$, then we are done.

(4) $\widehat{\xi}_t = \sum_{\Pi} | < X, Y >_{t_n+1} - < X, Y >_{t_n} |$. Note the following

$$| < X, Y >_{t_n+1} - < X, Y >_{t_n} | = \frac{1}{4} | < X + Y >_{t_{n+1}} - < X + Y >_{t_n} - < X - Y >_{t_{n+1}} + < X - Y >_{t_n} |$$
$$\leq \frac{1}{4} [\Delta_{t_n} < X + Y > + \Delta_{t_n} < X - Y >]$$
$$= \frac{1}{2} (< X >_{t_{n+1}} - < X >_{t_n}) + \frac{1}{2} (< Y >_{t_{n+1}} - < Y >_{t_n})$$

sum them up to get the desired inequality. □

Problem (1.5.11). *Let $\{X_t; \mathcal{F}_t, 0 \leq t < \infty\}$ be a continuous process with the property that for each fixed $t > 0$ and for some $p > 0$,*

$$\lim_{\|\Pi\| \to 0} V_t^{(p)} = L_t \quad (\textit{in probability})$$

where L_t is a random variable taking values in $\mathbb{R}^+$ a.s. Show that for $q > p$, $\lim_{\|\Pi\| \to 0} V_t^{(q)}(\Pi) = 0$ in probability, and for $0 < q < p$, the limit is infinite on the event $\{L_t > 0\}$.

Proof.

$$V^{(q)}(\Pi) = m_t(X)^{q-p} \sum_{k=1}^{n} |X_{t_{k+1}} - X_{t_k}|^p$$

where $m_t(X, \Pi) = \sup\{|X_t - X_s| : 0 \leq t, s \leq t, |s - t| \leq \|\Pi\|\}$, by uniformly continuity, this thing goes to zero a.s. Now the problem becomes obvious. □

Problem (1.5.12). *Let $X \in \mathcal{M}_2^c$, and T is a topping time of $\{\mathcal{F}_t\}$. If $< X >_T = 0$, a.s. $\mathbb{P}$, then we have $\mathbb{P}[X_{T \wedge t} = 0, \forall 0 \leq t < \infty] = 1$.*

Proof. $X_{t \wedge T}^2 - < X >_{T \wedge t}$ is a continuous martingale by one of the optional sample theorems, and since $< X >$ is an increasing and continuous, hence

$$\mathbb{P}[< X >_{t \wedge T} = 0; \forall 0 \leq t < \infty] = \sum_{r \in \mathbb{Q}^+} \mathbb{P}[< X >_{r \wedge T} = 0] = 0$$

□

Problem (1.5.14). *Show that for $X, Y \in \mathcal{M}_2^c$ and Π a partition of $[0, 5]$,*

$$\lim_{\|\Pi\| \to 0} \sum_{k=1}^{m} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) = < X, Y >_t \quad \textit{in probability}$$

Proof. Only thing we have to consider now is their difference:

$$\mathbb{E}\left[\sum_{k=1}^{m} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) - < X, Y >_t\right]^2$$
$$= \sum_{k=1}^{m} \mathbb{E}\left[(X_{t_k} - X_{t_{k-1}})^2 (Y_{t_k} - Y_{t_{k-1}})^2 - 2(X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}})(< X, Y >_{t_k} - < X, Y >_{t_{k-1}}) + (< X, Y >_{t_k} - < X, Y >_{t_{k-1}})^2\right]$$
$$+ \mathbb{E}[\sum_{1 \leq k \neq l \leq m} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) - (< X, Y >_{t_k} - < X, Y >_{t_{k-1}}))((X_{t_l} - X_{t_{l-1}})(Y_{t_l} - Y_{t_{l-1}}) - (< X, Y >_{t_l} - < X, Y >_{t_{l-1}}))$$

use conditional expectation we see that the second term is zero. Assume for now Y, X be bounded by K, and consider the first sum, all terms goes to zero by versions of Holder's inequality and bounded convergence theorem. If X, Y are not bounded, then using $T_n = \{\max\{|X_t|, |Y_t|\} \geq n\}$ and localization. $\square$

Problem (1.5.17)**.** *Let $X, Y \in \mathcal{M}^{c,loc}$. Then there is a unique (up to indistinguishability) adapted, continuous process of bounded variation $< X, Y >$ satisfiying $< X, Y >_0 = 0$ a.s. $\mathbb{P}$, such that $XY - < X, Y > \in \mathcal{M}^{c,loc}$. If $X = Y$, we write $< X > = < X, Y >$, and this process is nondecreasing.*

Proof. By definition, there exists an nondecreasing sequence of stopping times such that $T_n \to \infty$ a.s. $\mathbb{P}$ such that $X_{t\wedge T_n}, Y_{t\wedge T_n}$ are Mtg's. So denote $< X, Y >_t^{(n)}$ be the cross variation of $X_t^{(n)} Y_t^{(n)} = (XY)_{t\wedge T_n}$, meaning $(XY)_{t\wedge T_n} - < X, Y >_t^{(n)}$ is a martingale.

Now note that $(XY)_{t\wedge T_n \wedge T_{n-1}} - < X, Y >_{t\wedge T_{n-1}}^{(n)} = (XY)_{t\wedge T_{n-1}} - < X, Y >_t^{(n-1)}$ a.s. $\mathbb{P}$ by uniqueness of cross-variation. So $< X, Y >_t^{n-1} = < X, Y >_t^n$ on $\{t \leq T_{n-1}\}$, so define $< X, Y >_t (\omega) \triangleq < X, Y >_t^{(n)}$ for $T_n \geq t$ a.s. and it is an increasing process. $\square$

Problem. *1.5.19*

1. *A local martingale of class DL is a martinagle.*
2. *A nonnegative local martingale is a super-martingale.*
3. *If $M \in \mathcal{M}^{c,loc}$ and S is a stopping time of $\mathcal{F}_t$, then $E(M_S^2) \leq E < M >_S$, where $M_\infty^2 \triangleq \liminf_{t\to\infty} M_t^2$.*

Proof. (1) Suppose X is a local mtg of class DL, meaning for family $\mathcal{L}_a$ and $T \in \mathcal{L}_a(\mathbb{P}[T \leq a] = 1$ for some fixed number $a > 0)$,$\{X_T\}_{T\in\mathcal{L}_a}$ is uniformly integrable. First of all, there exists $\{T_n\}_{n\in\mathbb{N}}$ where $\mathbb{P}[T_n \to \infty] = 1$ such that $X_{T_n \wedge t}$ is a martingale for all n. Now, choose any $S < T$ bounded sotpping times, they are in some class $\mathcal{L}_a$ for some a. So by optional sampling theorem, $\mathbb{E}[X_{T\wedge T_n}] = \mathbb{E}[X_{S\wedge T_n}]$, now take n large so $a < T_n$ a.s. $\mathbb{P}$, so we have $\mathbb{E}[X_T] = \mathbb{E}[X_S]$ for all $S \leq T$, and same in reverse order, so by problem **1.3.26**, X is a martingle.

(2) Since nonnegative, we can use Fatou's Lemma: Let S be a bounded stopping time of $\mathcal{F}_t$, then by Fatou's Lemma we have $\mathbb{E}[\liminf X_{t\wedge T_n \wedge S} | \mathcal{F}_s] \leq \liminf X_{s\wedge T_n \wedge S}$. Now, since T_n increases to ∞ almost surely, $\liminf_{t\to\infty} X_{t\wedge T_n \wedge S} = X_{t\wedge S}$ a.s. $\mathbb{P}$, so let $t > S$, and by **Problem 1.3.26** again, we are done.

(3) So far we have $\mathbb{E}[M_{t\wedge T_n}^2] = \mathbb{E}[< X >_{t\wedge T_n}]$ for all n and t, and $\mathbb{E}[M_{t\wedge T_n \wedge S}^2] = \mathbb{E}[< X >_{t\wedge T_n \wedge S}]$. Now, take $\liminf$ and use Fatou's lemma again to get

$$\mathbb{E}[\liminf_{n\to\infty} X_{t\wedge T_n \wedge S}^2] \leq \liminf \mathbb{E}[< X >_{t\wedge T_n \wedge S}]$$

Note that $< X >_{t\wedge T_n}$ is nondecreasing both in t and n since T_n is increasing. So by Monotone convergence theorem, we can move the $\liminf$ inside of the expectation, and since T_n eventually increases to infinity, we have

$$\mathbb{E}[X_{t\wedge S}^2] \leq \mathbb{E}[< X >_{t\wedge S}]$$

now use Fatou's lemma and the fact that the quadratic variation of a local martingale is nondecreasing and monoton convergence theorem again

$$\mathbb{E}[\liminf_{t\to\infty} X_{t\wedge S}^2] \leq \mathbb{E}[< X >_S]$$

Then the conclusion holds true on the set $\{S < \infty\}$, and taking into account that M_∞^2 exists (in the sense of lim inf), then we are done. $\square$

Excercise (1.5.20)**.** *Suppose $X \in \mathcal{M}_2$ has stationary, independent increments. Then $< X >_t = t(\mathbb{E}X_1^2)$.*

Proof. Say $t < 0$, then $X_t^2 - < X >_t$ is a mtg starting with zero. Also we have $X_t - X_s \sim X_{t-s}$ for $t \geq s$, and $X_t - X_s \perp X_u - X_v$ for $v \leq u \leq s \leq t$.

$$\begin{aligned} \mathbb{E}[X_t^2 - X_s^2 - (t-s)\mathbb{E}X_1^2 | \mathcal{F}_s] &= \mathbb{E}[(X_t - X_s)^2] - (t-s)\mathbb{E}X_1^2 \\ &= \mathbb{E}[X_{t-s}^2] - (t-s)\mathbb{E}X_1^2 \end{aligned}$$

Now, let $f(t) = \mathbb{E}[X_t^2]$, then we have $f(t-s) = f(t) - f(s)$, the only solution to this equation is $f(x) = cx$, where $c = f(1)$ (I actually did not know this). $\square$

Excercise (1.5.21). *Employ the localization technique used in the solution of problem 5.17 to establish the following extension of problem 5.12: If $X \in \mathcal{M}^{c,loc}$ and for some stopping time T of $\mathcal{F}_t$ we have $< X >_T = 0$ a.s. $\mathbb{P}$, then $\mathbb{P}[X_{T\wedge t} = 0; \forall 0 \le t < \infty] = 1$. In particular, every $X \in \mathcal{M}^{c,loc}$ of bounded first variation is identically equal to zero.*

Proof. We have $X_{t\wedge T_n}$ is a martinagle for all T_n stopping time $\mathcal{F}_t$ and $T_n \uparrow \infty$ a.s., then we'd have $< X >_{T_n\wedge T} = 0$ since $< X >$ is nondecreasing, so $X_{t\wedge T_n}$ is identically zero for all $n \in \mathbb{N}$, so take $n \to \infty$. □

Problem (1.5.24). *Let $M \in \mathcal{M}_2 \cup \mathcal{M}^{c,loc}$ and assume that its quadratic variation process $< M >$ is integrable: $\mathbb{E} < M >_\infty < \infty$. Then*

1. *M is a martinagle, and M and the submartingale M^2 are both uniformly integrable; in particular, $M_\infty = \lim_{t\to\infty} M_t$ exists a.s. $\mathbb{P}$, and $\mathbb{E}[M_\infty^2] = \mathbb{E} < M >_\infty$.*
2. *We may take a right-continuous modification of $Z_t = \mathbb{E}[M_\infty^2|\mathcal{F}_t] - M_t^2; t \ge 0$, whic is a potential.*

Proof. (1) Assume $M \in \mathcal{M}^{c,loc}$, we have from **problem 1.5.19** that $\mathbb{E}[M_S^2] \le \mathbb{E}[< M >_S] le \mathbb{E}[< X >_\infty]$ for all stopping time of $\mathcal{F}_t$, that includes the case $S = t$ for $t \in \mathbb{R}^+$, then by Durrett, 2019, $\{M_S\}_{S\in\mathcal{L}}$ is uniformly integrable, hence of class DL, and by **Problem 1.5.19** (i), it is a martinagle.

Now by the uniform integrability of M, we know that $M_t \to M_\infty$ for some M_∞ integrable and in L^2

$$\mathbb{E}[\lim_{t\to\infty} M_t^2] \le \lim_{t\to\infty} \mathbb{E}[M_t^2] \le \mathbb{E}[< M >_\infty]$$

so we have L^1 convergence of nonnegative submartingale, so by one of the mtg convergence theorem, M_t^2 is also uniformly integrable, moreover, M_∞^2 is its last element. So $Z_t \ge 0$ a.s. $\mathbb{P}$, and $\mathbb{E}[Z_t] = \mathbb{E}[M_\infty^2 - M_t^2] \to 0$ as $t \to \infty$. □

Problem (1.5.25). *let $M \in \mathcal{M}^{c,loc}$ and show that for any stopping time T of $\mathcal{F}_t$,*

$$\mathbb{P}\left\{\max_{0\le t\le T} |M|_t \ge \epsilon\right\} \le \frac{\mathbb{E}[\delta\wedge < M >_T]}{\epsilon^2} + \mathbb{P}[< M >_T \ge \delta]$$

$\forall \epsilon, \delta > 0$. In particular, for a sequence $\{M^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{M}^{c,loc}$ we have

$$< M^{(n)} >_T \to_{\mathbb{P}} 0 \quad \implies \quad \max_{0\le t\le T} |M_t^{(n)}| \to_{\mathbb{P}} 0$$

Proof. By **1.5.19 (ii)** we have $\mathbb{E}[M_T^2] \le \mathbb{E}[< M >_T]$ holds for any bounded stopping time T of $\mathcal{F}_t$. The n by **Remark 1.4.17** we have the desired conclusion. □

Problem (1.5.26). *Let $M, N \in \mathcal{M}^{c,loc}$ with $\mathcal{F}_t$ and $\mathcal{H}_t$ as filtrations respectively, and suppose $\mathcal{F}_\infty \perp \mathcal{H}_\infty$. With $\mathcal{G}_t = \triangle\sigma(\mathcal{F}_t \cup \mathcal{H}_t)$ show that M, N, MN are all local martinagles with repsect to $\mathcal{G}_t$.*

2. Brownian Motion

Problem (2.1.4). *Let X be a stochastic process for which $X_0, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are indepdnent random variables, for every integer $n \ge 1$ and partition $\{t_i\}_{1\le i\le N}$ of the real line. Show that for any fixed $0 \le s < t < \infty$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s^X$.*

Proof. $\sigma_{\Pi_n} = \sigma(X_{t_1}, ..., X_{t_n}) = \sigma(X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}})$, and they are all subset of the collection of the set that is independent of $X_t - X_s$, call it $\mathcal{D}$, which is a Dynkin's system, and call the collection of all σ_{Π_n} $\mathcal{G}$, then $\mathcal{G} \subset \mathcal{D}$, and by the Dynkin's system theorem, we are done. □

2.4 The Space $C[0, \infty)$, Weak Convergence, and the Wiener Measure

Define

$$\rho(\omega_1, \omega_2) \triangleq \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0\le t\le n} (\|\omega_1(t) - \omega_2(t)\| \wedge 1)$$

The metric on the space $C[0, \infty)$

Problem (2.4.1). *Show that ρ above is a metric on $C[0, \infty)$ and under ρ, the space is complete and seperable metric space.*

Proof. Metric: Obvious. Suppose $f_n \to f$ in ρ, that is

$$\sum_{n\in\mathbb{N}} \frac{1}{2^n} \max_{1\le t\le n} (|f_n(t) - f(t)| \wedge 1) \to 0$$

so each $\max_{1\le t\le n}(|f_n(t) - f(t)|) \to 0$. Suppose $x \in [0,\infty)$, then $0 \le x \le n$ for some n, then we have uniformly convergence on $[0,n]$ of f_n to f, hence f continous at x. □

Problem (2.4.2). *Let $G(G_t)$ be collection of finite-dimensional cylinder sets of the for*

$$C = \{\omega \in C[0,\infty) : (\omega(t_1), ..., \omega(t_n)) \in A\} \quad n \ge 1, A \in \mathcal{B}(\mathbb{R}^n)$$

where, for all $i = 1, ..., n, t_i \in [0,\infty)$(respectively, $t_i \in [0,t]$). Denote by $\mathcal{G}(\mathcal{G}_t)$ the smallest σ-field containing $G(G_t)$.

Show that $\mathcal{G} = \mathcal{B}(C[0,\infty))$, the borel σ-field generated by the open sets in $C[0,\infty)$, and that $\mathcal{G}_t = \varphi_t^{-1}(\mathcal{B}(C[0,\infty)) \triangleq \mathcal{B}_t(C[0,\infty))$, where $\varphi_t : C[0,\infty) \to C[0,\infty)$ is the mapping $\varphi_t(\omega)(s) = \omega(t \wedge s)$ for $0 \le s < \infty$.

Proof. I guess open sets are defined by the metric ρ. Note that $\mathcal{G}$ is generated by $H = \{\omega \in C : \omega(t) \in A\}$ where $t \in \mathbb{R}^+$ and $A \in \mathcal{B}(\mathbb{R})$, or we can even take A to be open. For all $\omega \in H$, H contians $B_\epsilon(\omega)$ with ϵ small enough, so $\mathcal{G} \subset \mathcal{B}(C[0,\infty))$, so H is open in ρ. Since we working with continuous functions, the summants in ρ can be taken to be the sup of rationals, so we have the other direction.

For the second part: Let C be a cylinder set

$$\begin{aligned}\varphi_t^{-1}(C) &= \{\omega \in C([0,\infty)) : (\varphi(\omega)(t_i))_{1\le i\le n} \in A\}\\ &= \{\omega \in C([0,\infty)) : (\omega(t_i \wedge t))_{1\le i\le n} \in A\} \in \mathcal{G}_t\end{aligned}$$

If $C \in \sigma(G_t)$, we can take it to be $\{\omega \in C[0,\infty) : (\omega(t_i))_{1\le i\le n} \in A\}$ where $t_i \in [0,t]$ for all i and $A \in \mathcal{B}(\mathbb{R}^n)$, then $C = \{\omega \in C[0,\infty) : (\varphi\omega(t_i))_{1\le i\le n} \in A\} \in \mathcal{B}_t(C[0,\infty))$. □

Problem (2.4.5). *Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables taking values in a metric space (S_1, ρ_1) and converging in distribution to X and suppose (S_2, ρ_2) is another metric space, and $\varphi : S_1 \to S_2$ is continous. Show that $Y_n \triangleq \varphi(X_n)$ converges in distributiion to $Y \triangleq \varphi(X)$.*

Proof. Let f be a continous function on S_2, and consider

$$\mathbb{E}[f(\varphi(X_n))] = \mathbb{E}[f \circ \varphi(X_n)] \to \mathbb{E}[f \circ \varphi(X)]$$

by definition. □

Def. *Let Π be a family of probability measure, it is relatively compact if every sequence of elements of Π contains a weakly convergent subsequence. It is tight if for every $\epsilon > 0$, there exists compact set $K \subset S$ such that $\mathbb{P}[K] \ge 1 - \epsilon$ for all $\mathbb{P} \in \Pi$.*

tight is similar to equicontinuous

Theorem (Prohorov). *Let Π be a family of probability measures on a complete seperable metric space S. Π is relatively compact if and only if it is tight.*

Def. *If $\omega \in C[0,\infty)$ and $\delta > 0$, we define modulus of continuity on $[0,T]$ as*

$$m^T(\omega,\delta) \triangleq \max_{|s-t|<\delta; 0\le s,t\le T} |\omega(s) - \omega(t)|$$

Problem (2.4.8). *Show that $m^T(\omega,\delta)$ is continous in $\omega \in C[0,\infty)$ under the metric ρ as abov, is nondecreasing in δ, and $\lim_{\delta\downarrow 0} m^T(\omega,\delta) = 0$ for each $\omega \in C[0,\infty)$.*

Proof. First for continuity: Let

$$\lim_{n\to\infty} \rho(\omega_n, \omega) = 0 \Longleftrightarrow \lim_{n\to\infty} \sum_{1\le k<\infty} \frac{1}{2^n} \left(\max_{0\le t\le k} |\omega_n(t) - \omega(t)| \wedge 1 \right) \to 0$$

In particular, $\omega_n \to \omega$ uniformly on $[0,T]$. So consider

$$\begin{aligned}|m^T(\omega,\delta) - m^T(\omega_n,\delta)| &= \left| \max_{|s-t|<\delta; 0\le s,t\le T} |\omega(s) - \omega(t)| - \max_{|s-t|<\delta; 0\le s,t\le T} |\omega_n(s) - \omega_n(t)| \right|\\ &\le \max_{|s-t|<\delta; 0\le s,t\le T} |\omega(s) - \omega(t) - \omega_n(s) + \omega_n(t)|\end{aligned}$$

It goes to zero by triangular inequality and relaxing the restriction for which we are taking maximum of.

The second assersion is true due to uniform continuity. □

Theorem (Arzala-Ascoli)**.** *A set $A \subset C[0,\infty)$ has compact closure if and only if the following two condition holds:*

$$\sup_{\omega\in A} |\omega(0)| < \infty$$
$$\lim_{\delta\downarrow 0} \inf \sup_{\omega\in A} m^T(\omega,\delta) = 0 \quad \textit{for every } T > 0$$

Problem (2.4.11)**.** *Let $\{X^{(m)}\}_{m\in\mathbb{N}}$ be a sequence of continuous stochastic processes $X^{(m)} = \{X_t^{(m)}; 0 \le t < \infty\}$ ono $(\Omega,\mathcal{F},\mathbb{P})$, satisfying the following conditions*

- $\sup_{m\geq 1} \mathbb{E}\left|X_0^{(m)}\right|^\nu \triangle M < \infty$
- $\sup_{m\geq 1} \mathbb{E}\left|X_t^{(m)} - X_s^{(m)}\right|^\alpha \le C_T|t-s|^{\beta+1}; \quad \forall T > 0$ *and* $0 \le s,t \le T$

for some positive constant α,β,ν universally and C_T depending on $T > 0$.

Show that the probability measure $\mathbb{P}_m = \mathbb{P}(X^{(m)})^{-1}; m \geq 1$ induced by these processes on $(C[0,\infty),\mathcal{B}(C[0,\infty)))$ form a tight sequences.

Proof.

$$\mathbb{P}[|X_0^{(m)}| \geq \lambda] \le \frac{\mathbb{E}|X_0^{(m)}|^\nu}{\lambda^\nu} \le \frac{M}{\lambda^\nu} \quad \text{uniformly}$$

so taking sup and take $\lambda \to \infty$ this thing converges to zero, which verifies condition (4.6) of **Theorem 4.10** since $\mathbb{P}[|X_0^{(m)}| \geq \lambda] = \mathbb{P}_m(|\omega(0)| \geq \lambda)$ since we are assuming those are coordinate mapping processes.

Now let $m^T(\omega,\delta) \triangleq \max_{|t-s|<\delta, 0\le s,t\le T} |\omega(s) - \omega(t)|$. Again by Chebyshev we have

$$\mathbb{P}\left[|X_{\frac{k}{2^n}} - X_{\frac{k+1}{2^n}}| \geq \epsilon_n\right] \le \frac{\mathbb{E}\left[|X_{\frac{k}{2^n}} - X_{\frac{k+1}{2^n}}|^\alpha\right]}{\epsilon_n^\alpha} \le \epsilon_n^{-\alpha} C_T 2^{-(n\beta+n)}$$

So let $D_n \triangleq \{r \in [0,1] : r = \frac{k}{2^n} \text{ for some } k \text{ and } n\}$, then $\cup_{n\geq 1} D_n$ would be the set of dyadic rationals dense in $[0,1]$, let $\widetilde{D}_n = \bigcup_{k\geq n} D_k$. Also let $\epsilon_n = 2^{-n}$ and get,

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le 2^n} \left|X_{\frac{k}{2^n}} - X_{\frac{k+1}{2^n}}\right| > \epsilon_n\right) &= \mathbb{P}\left(\bigcup_{2^n\geq k\geq 1} \left|X_{\frac{k}{2^n}} - X_{\frac{k+1}{2^n}}\right| > \epsilon_n\right) \\
&\le \sum_{k=1}^{2^n} \mathbb{P}\left(\left|X_{\frac{k}{2^n}} - X_{\frac{k+1}{2^n}}\right| > \epsilon_n\right) \\
&\le \epsilon_n^{-\alpha} C_T 2^{-n\beta}
\end{aligned}$$

Note that those forms a summable series, so by Borel-Cantelli, we have

$$\mathbb{P}\left(\bigcap_{n\geq 1}\bigcup_{k\geq n} \max_{1\le i\le 2^k} \left|X_{\frac{i}{2^k}} - X_{\frac{i+1}{2^k}}\right| > \epsilon_n\right) = 0$$

That is, for all $\delta > 0$, there eixsts $n \geq 1$ such that]

$$\mathbb{P}\left(\bigcup_{k\geq n} \max_{1\le i\le 2^k} \left|X_{\frac{i}{2^k}} - X_{\frac{i+1}{2^k}}\right| > \epsilon_n\right) = \mathbb{P}\left(\max_{|s-t|\le\epsilon_k; s,t\in\widetilde{D}_k} |X_t - X_s|\right) < \delta$$

where in the proof we use X to denote arbitrary $X^{(m)}$, and since we can take sup on each probability measure, so we are done. □

Problem (2.4.12)**.** *Suppose $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ is a sequence of probability measures on $(C[0,\infty),\mathcal{B}(C[0,\infty)))$ which converges weakly to a probability measurable $\mathbb{P}$. Suppose, in addition, that $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of real valued continuous function on $C[0,\infty)$ converging to a continuous function f, the convergence being uniform on compact subsets of $C[0,\infty)$. Then*

$$\lim_{n\to\infty} \int_{C[0,\infty)} f_n(\omega) d\mathbb{P}_n(\omega) = \lim_{n\to\infty} \int_{C[0,\infty)} f(\omega) d\mathbb{P}(\omega)$$

Proof. Let B_n be the closed ball of radius $n \in \mathbb{N}$ in $C[0,\infty)$, then $\forall \epsilon > 0, \exists N \geq 1$ such that $\mathbb{P}(B_n^c) < \epsilon$ for all $n \geq N$.

$$\begin{aligned}
&\left|\int_{C[0,\infty)} f_n d\mathbb{P}_n - \int_{C[0,\infty)} f d\mathbb{P}\right| \\
=&\left|\int_{C[0,\infty)} f_n d\mathbb{P}_n - \int_{C[0,\infty)} f_n d\mathbb{P} + \int_{C[0,\infty)} f_n d\mathbb{P} - \int_{C[0,\infty)} f d\mathbb{P}\right| \\
\leq&\left|\int_{C[0,\infty)} f_n d\mathbb{P}_n - \int_{C[0,\infty)} f_n d\mathbb{P}\right| + \left|\int_{C[0,\infty)} f_n d\mathbb{P} - \int_{C[0,\infty)} f d\mathbb{P}\right|
\end{aligned}$$

Now doubt the second term goes to zero, now look at the first one

$$\begin{aligned}
&\left|\int_{C[0,\infty)} f_n d\mathbb{P}_n - \int_{C[0,\infty)} f_n d\mathbb{P}\right| \\
\leq&\left|\int_{C[0,\infty)} f_n d\mathbb{P}_n - \int_{C[0,\infty)} f d\mathbb{P}_n\right| + \left|\int_{C[0,\infty)} f d\mathbb{P}_n - \int_{C[0,\infty)} f_n d\mathbb{P}\right| \\
\leq& 2M\epsilon + \left|\int_{C[0,\infty)} f_n \varphi_k d\mathbb{P}_n - \int_{C[0,\infty)} f \varphi_k d\mathbb{P}_n\right| + \left|\int_{C[0,\infty)} f \varphi_k d\mathbb{P}_n - \int_{C[0,\infty)} f_n \varphi_k d\mathbb{P}\right|
\end{aligned}$$

where $0 \leq \varphi_k \leq 1$ takes value 1 on B_k and zero on $B_k(1+\delta)$ for some small δ given by Urysoln's Lemma and M be the uniform bound of f_n and as well as f. Then clearly second term goes to zero. Now look at the first term. Since we have locally uniform convergence, let n so large that $\|f_n - f\|_{L^\infty(B_k)} < 2^{-N}$, then the integral would be less than 2^{-N} as well, so it converges to zero. □

2.5. The Markov Property

Problem (2.5.2)**.** *Show that for each $F \in \mathcal{B}(C[0,\infty])$, the mapping $x \mapsto P^x(F)$ is $\mathcal{B}(\mathbb{R}^n)\backslash\mathcal{B}([0,1])$-measurable.(Hint: Dynkin System)*

Proof. Observations: Let $\mathcal{D} \triangleq \{A \in \mathcal{B}(C[0,\infty))$ for which the map $x \mapsto P^x(A)$ is $\mathcal{B}(\mathbb{R}^n)\backslash\mathcal{B}([0,1])$ measurable$\}$. Suppose $A, b \in \mathcal{D}$ and $A \subset B$, then $x \mapsto P^x(B) - P^x(A) = P^x(B-A)$ is $\mathcal{B}(\mathbb{R}^n)\backslash\mathcal{B}([0,1])$ measurable, same for an increasing sequence A_n. So $\mathcal{D}$ is a Dynkin's system. Now we only need to show that the generating sets of $\mathcal{B}(C[0,\infty))$ is in $\mathcal{D}$. So by **Problem 2.4.2**, we only need to show this for $A'\{\omega \in C[0,\infty) : \omega(t) \in A\}$ for arbitrary $t \geq 0$. So

$$x \mapsto P^0(\{\omega \in A + x\}) = \int_{A+x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{\xi^2}{2t^2}} d\xi = \int_A \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\xi+x)^2}{2t^2}} d\xi$$

turns out to be a continous function, so measurable. □

Problem (2.5.4(3))**.** *The coordinate mapping process $B = \{B_t, \mathcal{B}_t^B; t \geq 0\}$ oon $\left(C[0,\infty)^d, \mathcal{B}\left(C[0,\infty)^d\right), \mathbb{P}^\mu\right)$ is a d-dim Brownian Motion with intial distributed μ.*

Proof. It is definitely adapted and continous.

$$\begin{aligned}
\mathbb{P}^\mu(B_0 \in \Gamma) &= \int_{\mathbb{R}^d} \mathbb{P}^x(\Gamma)\mu(dx) \\
&= \int_{\mathbb{R}^d} \mathbb{P}^0(\Gamma - x)\mu(dx) \\
&= \int_{\mathbb{R}^d} \mathcal{X}_\Gamma(x)\mu(dx) = \mu(\Gamma)
\end{aligned}$$

and $B_t - B_s$ when $s < t$ is a Gassian vector in $\mathbb{R}^d$, so it is a Brownian motion with initial distribution μ. □

Problem. *2.5.5 Let $\{B_t = (B_t^i)_{1 \leq i \leq d}, \mathcal{F}_t, 0 \leq t < \infty\}$ be a d-dim Brownian Motion. Show that the process*

$$M_t^{(i)} \triangleq B_t^{(i)} - B_0^{(i)}, \mathcal{F}_t; \quad 0 \leq t < \infty, 1 \leq i \leq d$$

are continous, square integrable martingales, with $< M^{(l)}, M^{(j)} >_t = t\delta_{lj}$. Furthermore, the vector of martinagles $M = (M^{(i)})_{1 \leq i \leq d}$ is indepdnent of $\mathcal{F}_0$.

Obvious

Def (2.5.6)**.** *Let (S,ρ) bea metric space, we denote by $\overline{\mathcal{B}(\mathcal{S})}^{\mu}$ the complition of the Borel σ field(generated by the open sets) with respect to the finite measure μ on the metric space. The universal σ-field is $\mathcal{U}(S) \triangleq \bigcap_{\mu} \overline{\mathcal{B}(S)^{\mu}}$, where the intersection is over all finite measures, or equivalently, all probability measures, μ. A $\mathcal{U}(S)/\mathcal{B}(\mathbb{R})$-measurable, real valued function is said to be universally measurable.*

Problem (2.5.7)**.** *Let (S,ρ) be a metriic space and let f be a real-valued function defined on S. Show that f is universally measurable if and only if for every finite measurable μ on $(S,\mathcal{B}(S))$, there is a borel measurable function $g_{\mu} : S \to \mathbb{R}$ such that $\mu\{x \in S : f(x) \neq g(x)\} = 0$.*

Proof. $(\Rightarrow)$: Suppose f is universally measurable, then $f^{-1}(B) \subset \overline{\mathcal{B}^{\mu}}$ for all probability measures μ. Pick any μ, by the nature of the complition of σ field, there exists g_{μ} that agrees with f almost everywhere.

$(\Leftarrow)$: we have that $\mu(f^{-1}(B) \triangle g_{\mu}^{-1}(B)) = 0$ for all $B \subset \mathcal{B}(\mathbb{R})$, so $f^{-1}(B) \in \overline{\mathcal{B}(S)^{\mu}}$. □

2.5 B. Markov Processes and Markov Families

Problem (2.5.9)**.** *make the preceding discussion rigorous by proving the following result:*
If X, Y are d-dim random vectors on $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{G}$ is s sub-σ-field of $\mathcal{F}$, $X \perp \mathcal{G}$ and $Y \in \mathcal{G}$, then for eveyr $\Gamma \in \mathcal{B}(\mathbb{R}^d)$"

$$\mathbb{P}[X+Y \in \Gamma|\mathcal{G}] = \mathbb{P}[X+Y \in \Gamma|Y], \quad a.s.\ \mathbb{P};$$
$$\mathbb{P}[X+Y \in \Gamma|Y=y] = \mathbb{P}[X+y \in \Gamma], \quad for\ \mathbb{P}Y^{-1}\text{-}a.e.y \in \mathbb{R}^d$$

Proof. (a) In this case, we only have to show that $\mathbb{P}[X+Y \in \Gamma|\mathcal{G}] = \mathbb{E}[1_{X+Y\in\Gamma}|\mathcal{G}] \in \sigma(Y)$. Let $D = A \times B$ where $A, B \in \mathcal{B}(\mathbb{R}^n)$. Then

$$\mathbb{P}[(X,Y) \in D|\mathcal{G}] = \mathbb{E}[1_{\{X\in A\}}1_{\{Y\in B\}}|\mathcal{G}] = 1_{\{X\in A\}}\mathbb{E}[1_{\{Y\in A\}}]$$

Also we have $\sigma(\{A \times B : A, B \in \mathcal{B}(\mathbb{R}^n)\}) = \sigma(\mathbb{R}^d \times \mathbb{R}^d)$. Now let $D = \{(x,y) : x+y \in \Gamma\} \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$. Similar proof for the second one. □

3.2 Construction of Stochastic Integral

Problem (2.12)**.** *Let W be a standard one dim B-M, and let T be a stopping time of $\mathcal{F}_t$ of the BM with $\mathbb{E}[T] < 0$. Prove the Wald Identities*

$$\mathbb{E}[W_T] = 0; \quad \mathbb{E}[W_T^2] = \mathbb{E}[T]$$

Proof. Note that we must have $T < \infty$ a.s. $\mathbb{P}$, so we have $W_{t\wedge T}(\omega) \to W_T(\omega)$ a.s. pointwise (or this can be obtained from **Submartingale Convergence Theorem**), also $W^2_{t\wedge T}(\omega) \to W^2_T(\omega)$ a.s. From **Problem 1.3.24(a)** we know that $W^2_{T\wedge T}$ is a submartingale with respect to the same filtration. Then perhaps we can show it is uniformly integrable. So consider

$$\begin{aligned}
\mathbb{E}[\mathcal{X}_{|W_{T\wedge t}|\geq K}|W_{T\wedge}|]^2 &\leq \mathbb{P}[|W_{T\wedge t}| \geq K]\mathbb{E}[W^2_{T\wedge t}] \\
&= \mathbb{P}[|W_{T\wedge t}| \geq K]\mathbb{E}[t \wedge Tq] \\
&= \mathbb{P}[|W_{T\wedge t}| \geq K]\mathbb{E}[T] \\
&\leq \mathbb{E}[T]\frac{\mathbb{E}[W^2_{T\wedge t}]}{K^2} \\
&\leq \mathbb{E}[T]^2\frac{1}{K^2}
\end{aligned}$$

So, $\lim_{K\to\infty}\sup_{t\geq 0}\mathbb{E}[\mathbb{1}_{|W_{T\wedge t}|}|W_{T\wedge t}|] = 0$. So $W_{T\wedge t}$ is uniformly integrable, hence by one of the martingale convergence theorem, $\mathbb{E}[W_T] = 0$. □

Excercise (3.2.13)**.** *Let W be as in previous problem, let $b \in \mathbb{R}$, and let T_b be the passage time to b, that is $T_b = \inf_{t\geq 0}\{W_t = b\}$. Use the provious problem to show that for $b \neq 0$, we have $\mathbb{E}T_b = \infty$.*

Proof. $\mathbb{E}[W_{T_b}] = \mathbb{E}[b] = b \neq 0$, so the assumption of **Problem 2.12** must not hold, which is $\mathbb{E}[T] < 0$ □

Problem (3.2.18). *Let* $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\}$ *and* $N_t = \{N_t, \mathcal{F}_t, t \in [0,\infty)\}$ *be in* $\mathcal{M}_1^c$ *and suppose* $X \in \mathcal{L}(M)^*_\infty$ *and* $Y \in \mathcal{L}(N)^*_\infty$. *Show that the martingales* $I^M(X), I^N(Y)$ *are uniformly integrable and have last elements* $I^M_\infty(X), I^N_\infty(Y)$, *the corss-variation* $< I^M(X), I^N(Y) >_t$ *converges alsmost surely as* $t \to \infty$ *and*

$$\mathbb{E}[I^M_\infty(X) I^N_\infty(Y)] = \mathbb{E}[< I^M(X), I^N(Y) >_\infty] = \mathbb{E}\int_0^\infty X_t Y_t d < M, N >_t$$

In particular,

$$\mathbb{E}\left(\int_0^\infty X_t dM_t\right)^2 = \mathbb{E}\int_0^\infty X_t^2 d < M >_t$$

Proof. By definition we have

$$A \triangleq \mathbb{E}\int_0^\infty X_t^2 d < M >_t < \infty \quad B \triangleq \mathbb{E}\int_0^\infty Y_t^2 d < N >_t < \infty$$

For uniformly integrability, we consider

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{|I^M_t(X)|>K} |I^M_t(X)|\right] &\le \mathbb{P}\left(|I^M_t(X)| > K\right) \cdot \mathbb{E}[I^M_t(X)^2] \\
&\le \frac{1}{K^2}\mathbb{E}[I^M_t(X)^2]^2 \\
&= \frac{1}{K^2}\mathbb{E}\int_0^t X^2 d < M >_t \\
&\le \frac{1}{K^2}\left(\mathbb{E}\int_0^\infty X^2 d < M >_t\right)^2 = \frac{A^2}{K^2}
\end{aligned}$$

which goes to zero as $K \to \infty$, so uniformly integrable. So by **Problem 1.3.20**, it converges both in L^1 and $\mathbb{P} - a.s.$ to some $I^M_\infty(X)$ which can be viewed as the last element, and same for $I^N(Y)$.

Now, for $t \in \mathbb{R}^+$, let $\hat{\xi}_t$ be the total variation of $< M, N >_t$, we have

$$\begin{aligned}
\left|< I^M(X), I^N(Y) >_t\right| &= \left|\int_0^t X_s Y_s d < M, N >_s\right| \quad \mathbb{P} - a.s. \\
&\le \int_0^t |X_s Y_s| d\hat{\xi}_t \\
&\le \left(\int_0^t X_s^2 d < M >_s\right)^{\frac{1}{2}} \left(\int_0^t Y_s^2 d < N >_s\right)^{\frac{1}{2}} \\
&\le 10\left[\int_0^t X_s^2 d < M >_s + \int_0^t Y_s^2 d < N >_s\right] \\
&\le 10A + 10B
\end{aligned}$$

which is integrable. Now, apply **Proposition 2.4** to $X_t \mathbb{1}_{t \ge T}$ and same for Y, we have

$$\left|\int_T^{t+T} X_s Y_s d < M, N >_s\right| \le \left(\int_T^{T+t} X_s^2 d < M >_s\right)^{\frac{1}{2}} \left(\int_T^{T+t} Y_s^2 d < N >_s\right)^{\frac{1}{2}}$$

which goes to zero as $T \to \infty$ for any $t > 0$, so we have $\int_0^t X_s Y_s d < M, N >_s$ converges pointwise a.s. $\mathbb{P}$, and it is bounded by $10(A + B)$ and we have

$$\lim_{t\to\infty} \mathbb{E}[I^M_t(X) I^N(Y)_t] = \lim_{t\to\infty} \mathbb{E}[< I^M(X), I^N(Y) >_t] = \lim_{t\to\infty} \int_0^t X_s Y_s d < M, N >_s$$

by DCT we can move the limit inside the integral. For the last equality, replace $I^N(Y)$ by $I^M(X)$. □

Problem (3.3.10). *With* $\{Z_t; 0 \le t < \infty\}$ *as in* ***Example 3.3.9***, *set* $Y = \frac{1}{Z_t}$, $0 \le t < \infty$, *which is well defined because* $\mathbb{P}[\inf_{t \le t \le T} Z_t > 0] = \mathbb{P}[\inf_{0 \le t \le T} \xi_t > -\infty] = 1$. *Show that* Y *satisfies the stochastic differential equation*

$$dY_t = Y_t X_t^2 dt - Y_t X_t dW, \quad Y_0 = 1$$

Proof. $Y_t = \exp\left(-\int_0^t X_s dW_s + \frac{1}{2}\int_0^t X_s^2 ds\right)$ and we call $\xi_t = \int_0^t X_s dW_s + \frac{1}{2}\int_0^t X_s^2 dt$. Use Ito's formula

$$\begin{aligned}Y_t &= Y_0 - \int_0^t Y_s d\xi_t + \frac{1}{2}\int_0^t Y_s d < \xi >_s + \frac{1}{2}\int_0^t Y_s X_s^2 ds \\ &= Y_0 - \int_0^t Y_s X_s dW_s + \int_0^t Y_s X_s^2 ds\end{aligned}$$

□

Problem (3.3.12). *Suppose we have two continuous semimartingales*

$$X_t = X_0 + M_t + B_t; \quad Y_t = Y_0 + N_t + C_t$$

where $M, N \in \mathcal{M}^{c,loc}$ and B, C are adapted, continuous processes of bounded variation with $B_0 = C_0 = 0$ a.s.. Prove the ***Integration by Parts Formula***

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - < M, N >_t$$

Proof. Here we apply **Theorem 3.3.6** to $f(t, x, y) = x \cdot y$. So we have

$$\begin{aligned}X_t Y_t = f(t, X_t) &= X_0 Y_0 + \int_0^t Y_s dB_t + \int_0^t X_s dC_t + \int_0^t Y_s dN_s + \int_0^t X_s dM_s + 2 \times \frac{1}{2}\int_0^t 1 d < M, N >_s \\ &= X_0 Y_0 + \int_0^t Y_s dX_s + \int_0^t X_s dY_s + < M, N >_t\end{aligned}$$

□

3.3 B. Martingale Characterization of Brownian Motion

3.3. C Bessel Processes, Questions of Recurrence

Problem (3.3.20). *Show that for each $d \geq 2$, the Bessel family with dimension d is a strong Markov family.*

Proof. Compare to **Definition 2.6.3**, (a) and (b) are included in $\mathbb{P}^x$. The rest is fairly obvious due to the connection between $\mathbb{P}$ and $\hat{\mathbb{P}}$. □

3.3.D Martingale Moment Inequalities

Excercise (3.3.25). *With W be a standard one dim BM and X be measurable adapted process satisfying*

$$\mathbb{E}[\int_0^T |X_t|^{2m}] dt < \infty$$

for some real number $T > 0$ and $m \geq 1$, show that

$$\mathbb{E}\left|\int_0^T X_t dW_t\right|^{2m} \leq (m(2m-1))^m T^{m-1} \mathbb{E}\int_0^T |X_t|^{2m} dt$$

Proof. Let $M_t = \int_0^t X_s dW_s$ which is a continuous Mtg, and $f(t, x) = x^{2m}$, then by Ito's we have

$$\begin{aligned}\mathbb{E} M_t^{2m} &= \mathbb{E}\left(\int_0^t 2m M_t^{2m-1} X_s dW_s + m(2m-1)\int_0^t M_s^{2m-2} X_s^2 dt\right) \\ &= \mathbb{E}\left(m(2m-1)\int_0^t M_s^{2m-2} X_s^2 dt\right) \\ &= m(2m-1)\int_0^t \mathbb{E}\left[M_s^{2m-2} X_s^2\right] dt\end{aligned}$$

$\mathbb{E}\left[M_s^{2m-2} X_s^2\right] \leq \mathbb{E}\left[\mathbb{E}[M_t^{2m-2} | \mathcal{F}_s] X_s^2\right] = \mathbb{E}\left[M_t^{2m-2} X_s^2\right]$ and again,

$$M_s^{2m-2} = 2(m-1)\int_0^t M_s^{2m-3} dW_s + (m-2)(m-3)\int_0^t M_s^{2m-4} X_s^2 ds$$

Then do induction. □

Excercise. *Let* $\{M = \left(M_t^{(1)}, ..., M_t^{(d)}\right), \mathcal{F}_t, 0 \le t < \infty\}$ *be a vector of continuous local Martinagle on some* $\{\Omega, \mathcal{F}, \mathbb{P}\}$*, and define*

$$A^{(i,j)} \triangleq < M^{(i)}, M_t^{(j)} >, \quad A_t(\omega) \triangleq \sum_{i,j=1}^{d} \check{A}^{(i,j)}(\omega)$$

where $\check{A}^{(i,j)}$ *denotes total variation of* $A^{(i,j)}$ *on* $[0,t]$*. Let* $T_s(\omega)$ *be the inverse of the function* $A_t(\omega) + t$*, i.e.* $A_{T(\omega)} + T_s(\omega) = s; 0 \le s < \infty$*.*

1. *Show that for each* s, T_s *is a stopping time of* $\mathcal{F}_t$*.*
2. *Define* $\mathcal{G}_s \triangleq \mathcal{F}_{T_s}; 0 \le s < \infty$*. Show that if* $\mathcal{F}_t$ *satisfies the usual condition, then so does* $\mathcal{G}_s$*.*
3. *Define*

$$N_s^{(i)} \triangleq M_{T_s}^{(i)}, \quad 1 \le i \le d; \quad 0 \le s < \infty$$

Show that for eafch $1 \le i \le d : N^i \in \mathcal{M}^{c,loc}$*, and the cross variation* $< N^i, N^j >_s$ *is an absolutely continuous function of* s *a.s.* $\mathbb{P}$*.*

Proof. (1) Consider $\{T_s \le t\} = \{A_t + t > s\} \in \mathcal{F}_t$.

(2) $\mathcal{G}_s = \mathcal{F}_{T_s} = \{A \in \mathcal{F} : A \cap \{T_s < t\} \in \mathcal{F}_t\}$, so $\mathcal{G}_0 = \mathcal{F}_0$ which contalls all $\mathbb{P}$ null sets. Now consider

$$\begin{aligned}\bigcap_{\epsilon>0} \mathcal{G}_t &= \bigcap_{\epsilon>0} \mathcal{F}_{T_{t+\epsilon}} \\ &= \bigcap_{\epsilon>0} \{A \in \mathcal{F} : A \cap \{T_{t+\epsilon} < s\} \in \mathcal{F}_s\} \\ &= \bigcap_{\epsilon>0} \{A \in \mathcal{F} : A \cap \{A_s + s > t + \epsilon\} \in \mathcal{F}_s\} \\ &= \mathcal{G}_s\end{aligned}$$

so right continuous.

(3) There is $\{S_n\}$ sequence of stopping time with $M_{S_n \wedge t} \in \mathcal{M}^c$. Note that T_s is finite a.s., so let $t < s$ and use Optional Sampling theorem we have

$$\mathbb{E}[M_{S_n \wedge T_s} | \mathcal{G}_t] = M_{S_n \wedge T_t}$$

□

3.5.C Continuous Local Martingale as Time-Changed Brownian Motion

Problem (3.4.5)**.** *Let* $A = \{A(t); 0 \le t < \infty\}$ *be a continuous, nondecreasing function with* $A(0) = 0, S \triangleq A(\infty) \le \infty$*, and define for* $0 \le s < \infty$

$$T(s) = \begin{cases} \inf\{t \ge 0 : A(t) > s\}; & 0 \le s < S \\ \infty; & s \ge S \end{cases}$$

Show that the function $T = \{T(s); 0 \le s < \infty\}$ *has the following properties*

1. T *is nondecreasing and right-continuous on* $[0, S)$*, with values in* $[0, \infty)$*. If* $A(t) < S; \forall t \ge 0$*, then* $\lim_{\uparrow S} T(s) = \infty$*.*
2. $A(T(s)) = s \wedge S; \quad 0 \le s < \infty$*.*
3. $T(A(t)) = \sup\{\tau \ge t; A(\tau) = A(t)\}; 0 \le t < \infty$*.*
4. *Suppose* $\varphi : \mathbb{R}^+ \to \mathbb{R}$ *is continous and has the property*

$$A(t_1) = A(t) \quad \text{for some } 0 \le t_1 < t \Rightarrow \varphi(t_1) = \varphi(t).$$

Then $\varphi(T(s))$ *is continous for* $0 \le s < S$*, and*

$$\varphi(T(A(t))) = \varphi(t); \quad 0 \le t < \infty$$

5. *For* $0 \leq t, s < \infty : s < A(t) \iff T(s) < t$ *and* $T(s) \leq t \Rightarrow t \leq A(t)$.

6. *If* G *is a bounded, measruable, real valued function or a nonnegative measurable, extended real valued function defined on* $[a,b] \subset [0,\infty)$, *then*

$$\int_a^b G(t)dA(t) = \int_{A(a)}^{A(b)} G(T(s))ds$$

Proof. (a) $\{t \geq 0 : A(t) > s_1\} \subset \{t \geq 0 : A(t) > s_2\}$ if $s_1 > s_2$, so the corresponding inf is also bigger in the first set. Now suppose $A(t) < S$ for $t \geq 0$, then $\lim_{t\to\infty} A(t) = S$. Since T is nondecreasing, suppose by contradiction that $T(\infty) = T_\infty < \infty$, then for all sequence $t_n \to S$ such that $\inf\{t \geq 0; A(t) > t_n\}$. Now take t_n be such that $A(t_n)$ is strictly increasing and $A(t_n) \to S$. Then $\inf\{t \geq 0; A(t) > A(t_n)\} = t_n$ which goes to infinity.

(b) Suppose $s < S$, then by continuity we have $T(s) = \sup\{t \geq 0 : A(t) \leq s\}$, then by continuity we are done. If $s \geq S$, then by definition we are also done.

(c) First suppose $A(t) = S$, then $T(A(t)) = \infty$, and since $A(t)$ is nondecreasing, we have $A(s) = S$ for all $s \geq t$, so $T(A(t)) = \infty = \sup\{\tau \geq 0 : A(\tau) = S\}$. Now suppose $A(t) < S$, then $T(A_t) = \inf I$, where $I \triangleq \{\tau \geq 0 : A(\tau) > A(t)\}$ exists. If $\tau > T(A(t))$ then $\tau \in I$, so $I = (T(A(t)), \infty)$ and $T(A(t)) = \sup(\mathbb{R}^+)\backslash I$.

(d) We have $T(A(t)) = t \wedge S$. Since $T(s)$ is right continous, we only need to show left continuity here. Let $t_n \to t^- < S$, we only need to look at the case where T is not continous at t. But here we have $A(T(t^-)) = A(T(t)) = t$, so $\varphi(T(t^-)) = \varphi(T(t))$, so still continuous.

(e) Suppose $0 \leq t, s < \infty$ and $s < A(t)$, where $T(s) = \inf\{\tau \geq 0 : A(\tau) > s\}$, there is a $0 < t' < t$ such that $s < A(t') < A(t)$, so $T(s) \leq t' < t$. Now suppose $T(s) < t$ meaning $t \in \{\tau \geq 0 : A(\tau) > s\}^\circ$. Now suppose $T(s) \leq t$, meaning $t \in \overline{\{\tau \geq 0 : A(\tau) > s\}}$.

(f) This is just the change of variable formula. □

Problem (3.4.7). *Show that if* $\mathbb{P}[S \triangleq < M >_\infty < \infty] > 0$, *it is still possible to define a Brownian Motion* B *for whcih* $M_t = B_{<M>_t}$ *holds.*

Proof. Let's say Ω is rich enough such that there is a independent Brownian Motion W. Define $S \triangleq \lim_{t\to\infty} < M >_t$. By **Problem 4.5** we have that

$$\{T(s) < t\} = \{s << M >_t\} \in \mathcal{F}_t \quad \{< M >_s \leq t\} = \{T(t) \geq s\}$$

Define $\mathcal{G}_t = \mathcal{F}_{T(t)}$, then $\{< M >_s \leq t\} \in \mathcal{G}_t$, so $< M >_s$ is a stopping time of $\mathcal{G}_t$. Also, recall a theorem says limit sup or inf of a sequence of stopping times is also a stopping time if the limit exists, so S is a stopping time of $\mathcal{G}_t$ as well. Now consider the martingale $\widetilde{M}_t = M_{t\wedge T(s)}$, there we have

$$< \widetilde{M} >_t = < M >_{t\wedge T(s)} \leq < M >_{T(s)} = s$$

So both $\widetilde{M}$ and $\widetilde{M}^2 - < \widetilde{M} >$ are uniformly integrable. Now let $s' < s$, then consider

$$\mathbb{E}[M_{T(s)} - M_{T(s')}|\mathcal{F}_{T(s')}] = 0$$
$$\mathbb{E}[\Big(M_{Ts} - M_{T(s')}\Big)^2 |\mathcal{F}_{T(s')}] = \mathbb{E}[< M >_{Ts} - < M >_{Ts'} |\mathcal{F}_{T(s')}]$$

so $\widetilde{M}_t$ is a square integrable martingale with respect to $\mathcal{G}_t \triangleq \mathcal{F}_{T(t)}$. Now let

$$B_t \triangleq W_t - W_{S\wedge t} + M_{T(t)}$$

We note that by the same argument as in the theorem we have $M_{T(t)}$ is almost surely continuous, now

$$< B >_t = t - S \wedge t + < M >_{T(t)} = t - S \wedge t + S \wedge t$$

os $< B >_t = t$ a.s.. □

3.5.C Girsanov Theorem

Problem (3.5.6). *Assume the hypothese of theorem 5.1 and suppose* Y *is a measurable adapted provess satisfying* $\mathbb{P}[\int_0^T Y_t^2 dt < \infty] = 1$; $0 \leq T < \infty$. *Under* $\mathbb{P}$ *we may define the Ito integral* $\int_0^t Y_s dW_s^{(i)}$, *whereas under* $\widetilde{\mathbb{P}}$ *we may define the Ito integral* $\int_0^t Y_s \widetilde{W}_s^{(i)}, 0 \leq t \leq T$. *Show that for* $1 \leq i \leq d$ *we have*

$$\int_0^t Y_s d\widetilde{W}_s^{(i)} = \int_0^t Y_s dW_s^{(i)} - \int_0^t Y_s X_s^{(i)} ds; \quad 0 \leq t \leq T, \quad \mathbb{P} \text{ and } \widetilde{P}. a.s.$$

Proof. Hint: Prop 2.24: $M \in \mathcal{M}^{c,loc}$ and X progressively measruable with $\int_0^t X_s^2 d < M >_s < \infty$ a.s.. Then $I^M(X)$ is the unique local martingale Φ such that for all $N \in \mathcal{M}^{c,loc}$ $< \Phi, N >_t = \int_0^t X_u d < M, N >_u$.

Now let's use **Proposition 5.4**: Let $N \in \mathcal{M}^{c,loc}$, and from **Prop 5.4** we know that

$$\widetilde{N}_t \triangleq N_t - \int_0^t X_s^{(i)} d < N, W_s^{(i)} >$$

is a local martinagle in $\widetilde{\mathcal{M}}_T^{c,loc}$. From **Prop 5.5** we know that every $\widetilde{N} \in \widetilde{\mathcal{M}}_T^{c,loc}$ has the above form. Now

$$< \int_0^t Y_s dW_s^{(i)} - \int_0^t Y_s X_s^{(i)} ds, \widetilde{N}_t > = < \int_0^t Y_s dW_s^{(i)}, N_t > \qquad \text{a.s.}\mathbb{P} \text{ and } \widetilde{\mathbb{P}}_T$$

Recall that $\widetilde{W}_t \triangleq W - \int_0^t X_s^{(i)} ds$, which is a standard Brownian Motion under $\widetilde{P}_T$. So consider

$$\begin{aligned} < \int_0^t Y_s d\widetilde{W}_s, \widetilde{N}_t > &= \int_0^t Y_s d < \widehat{W}, \widetilde{N} >_t \\ &= \int_0^t Y_s d < W, N >_t \quad \widetilde{\mathbb{P}}_T \text{ a.s.} \end{aligned}$$

□

Problem (3.5.7)**.** *Let T be a stopping time of the filtration $\{\mathcal{F}_t^W\}$ with $\mathbb{P}[T < \infty] = 1$. A neccessary and sufficient condition for the validity of Wald Identity is*

$$\mathbb{E}[\exp\left(\mu W_T - \frac{1}{2}\mu^2 T\right)] = 1$$

where μ is a given real number, that is

$$\mathbb{P}^{(\mu)}\left[T < \infty\right] = 1.$$

In particular, if $b \in \mathbb{R}$ and $\mu b < 0$, then this condition holds for the stopping time

$$S_b \triangleq \inf\{t \geq 0; W_t - \mu t = b\}$$

Proof. Let $Z(t) \triangleq \exp\left(\mu W_t - \frac{1}{2}\mu^2 t\right)$, then $Z(t)$ is a Martingale, and $\mathbb{P}^{(\mu)}$ is defined to be

$$\mathbb{P}^{(\mu)}(A) \triangleq \mathbb{E}[1_A Z(t)] \qquad \text{for } A \in \mathcal{F}_t^W$$

Then now consider

$$\begin{aligned} \mathbb{P}\left[T < \infty\right] &= \mathbb{P}\left[\bigcup_{n \geq 1}\{T < n\}\right] \\ &= \lim_{t \to \infty} \mathbb{P}\left[T < t\right] \\ &= \lim_{t \to \infty} \mathbb{E}\left[1_{\{T<t\}} Z(t)\right] \\ &= \lim_{t \to \infty} \mathbb{E}\left[1_{\{T<t\}} \mathbb{E}\left(Z(t) | \mathcal{F}_{t \wedge T}\right)\right] \\ &= \lim_{t \to \infty} \mathbb{E}\left(1_{\{T<t\}} Z(t \wedge T)\right) \\ &= \lim_{t \to \infty} \mathbb{E}\left[1_{\{T<t\}} Z(T)\right] = \mathbb{E}[Z(T)] \end{aligned}$$

where the last two equatilies are due to optional sampling and monotone convergence. This shows if and only if part of the statement. The second part of the problem is given directly by the discussion above the problem. □

Problem (4.5.8)**.** *Denote by*

$$h(t; b, \mu) \triangleq \frac{|b|}{\sqrt{2\pi t^3}} \exp\left[-\frac{(b - \mu t)^2}{2t}\right]; \quad t > 0, b \neq 0, \mu \in \mathbb{R}$$

*Use **Theorem 2.6.16** to show that*

$$h(\cdot; b_1 + b_2, \mu) = h(\cdot; b_1, \mu) * h(b\cdot; b_2, \mu); \quad b_1 b_2 > 0, \mu \in \mathbb{R}$$

where $*$ denote the convolutions.

Proof. **Theorem 2.6.16** says that for almost surely finite stopping time S, we have $B_{S+t} - B_S$ is a standard one dim Brownian Motion with respect to its nature filtration when B itself is a Brownian Motion.

Now, $h(t; b, \mu) = \frac{\mathbb{P}^{(\mu)}[T_b \in dt]}{dt}$ by R-N theorem. We know that $W_t \triangleq B_{T_{b_1}+t} - B_{T_{b_1}}$ is an standard one dim Brownian Motion that is independent of $\mathcal{F}_{T_{b_1}}$. Since since we have the condition $b_1 b_2 > 0$, then $T_{b_1+b_2} = T^W_{b_2}$, so we have

$$\mathbb{P}^\mu \left(T_{b_1+b_2} \in dt\right) = \mathbb{P}^\mu[T^W_{b_1} + T_{b_2} \in dt]$$

but we know that $T^W_{b_1} = T_{b_1}$ in measure. Hence we are done. □

Excercise (3.5.9)**.** *With $\mu > 0$ and $W_* \triangleq \inf_{t>0} W_t$, under $\mathbb{P}^\mu$ the random variable $-W_*$ is exponentially distributed with parameter 2μ, i.4.,*

$$\mathbb{P}^\mu [W_* \in db] = 2\mu e^{-2\mu b}, \quad b > 0$$

Proof. We look at

$$\begin{aligned}\mathbb{P}^\mu [-W_* < b] &= \mathbb{P}^\mu [T_{-b} = \infty] \\ &= 1 - \mathbb{P}^\mu [T_{-b} < \infty]\end{aligned}$$

then take the derivative we will get the desired answer. For this to work, $\mu b > 0$ must be satisfied. □

Excercise (3.5.11)**.** *Consider for $\nu > 0$ and $c > 1$, the stopping time of $\{\mathcal{F}^W_t\}$:*

$$R_c = \inf\left\{t \geq 0; \exp\left[\nu W_t - \frac{1}{2}\nu^2 t\right] = c\right\}$$

Show that

$$\mathbb{P}[R_c < \infty] = \frac{1}{c}, \quad \mathbb{E}^\nu R_c = \frac{2\log c}{\nu^2}$$

Proof. Let $Z_\nu(t) \triangleq \exp\left[\nu W_t - \frac{1}{2}\nu^2 t\right]$, and by **Problem 2.28** it is a martinagle. Now consider

$$\begin{aligned}\mathbb{P}^\mu [R_c < \infty] &\triangleq \mathbb{E}\left[\mathcal{X}_{\{R_c<\infty\}} Z(R_c)\right] \quad \text{same as before} \\ &= \mathbb{P}[R_c < \infty]c\end{aligned}$$

Now we only need to show that $\mathbb{P}^\mu [R_c < \infty] = 1$. Note that

$$\exp\left[\nu W_t - \frac{1}{2}\nu^2 t\right] = c \iff \nu\left(\widetilde{W}_t + \frac{1}{2}\nu t\right) = \log(c)$$

Where $\widetilde{W}_t$ is a one dim standard Brownian Motion under $\mathbb{P}^\mu$. By the assumption that $c > 1$, we see that $\mathbb{P}^\mu (R_c < \infty) = 1$.

Now for the second part, we use *Walds Identity*, we can use it because $Z(t)$ is not only a martinagle, it is also a positive super martinagle, therefore, by **Problem 1.3.68** this process has a last element Z_∞ and $\{Z_t; \mathcal{F}_t; 0 \leq t \leq \infty\}$ is a martinagle, hence by problem **1.3.19(20)** it is a uniformly integrable family of random variable, hence we can use *Walds*.

$$\mathbb{E}^\nu[\widetilde{W}_T] = 0 \Rightarrow 0 = \mathbb{E}[\widetilde{W}_t] = \frac{\log(c)}{\nu} - \frac{1}{2}\nu\mathbb{E}[T]$$

□

4 Brownian Motion and Partial Differential Equations

4.2 Harmonic Functions and the Dirichlet Problem

Problem (4.2.4). *Suppose D is a bounded and connected, u is defined and continuous on $\overline{D}$, and u is Harmonic in D. Show that u attains its maximum over $\overline{D}$ on ∂D. If v is another function, Harmonic in D and continous on $\overline{D}$, and $v = u$ on ∂D, then $v = u$ on D as well.*

Proof. By assumption, u is also Harmonic and continous on D°, so by the Maximum Principle, we know that the sup or max is accheived on the boudnary.

Now if $v = u$ on ∂D and they are both Harmonic, then $\omega \triangleq u - v$ is also Harmonic and $\omega \equiv 0$ on ∂D. So $\sup_D \omega = 0$. Also, $-\omega$ is also Harmonic and is zero on the boudnary, so $\sup_D -\omega = 0$ in D. so $\omega \equiv 0$ in D, hence $u = v$ on $\overline{D}$. ☐

Problem (4.2.16). *Let $D \subset \mathbb{R}^d$ be open, and suppose that $a \in \partial D$ has the property that there exists a point $b \neq a$ in $\mathbb{R}^2 \backslash D$, and a simple arc in $\mathbb{R}^d \backslash D$ connecting a to b. Show that a is regular.*

Proof. Using **Example 2.14** we can define a *barrier at* $(0, 0)$ if we assume $a = (0, 0)$ using the curve that connecting a and b as a slid. ☐

Problem (4.2.25). *Consider as given an open, bounded subset $D \subset \mathbb{R}^d$ and the bounded, continous function $g : D \to \mathbb{R}$ and $f : \partial D \to \mathbb{R}$. Assume that $u : \overline{D} \to \mathbb{R}$ is continuous, of class $C^2(D)$, and solvese the Poisson Equation*

$$\frac{1}{2}\Delta u = -g$$

subject to the boudnary condition

$$u = f; \quad \text{on } \partial D$$

Then establish the representation

$$u(x) = \mathbb{E}^x \left(f(W_{\tau_D}) + \int_0^{\tau_D} g(W_t)dt \right); \quad x \in \overline{D}$$

In particular, the expected exit time from a ball is given by

$$\mathbb{E}^x \left(\tau_{B_r} \right) = \frac{r^2 - \|x\|^2}{d}; \quad x \in B_r$$

Hint: Show that the process $\{M_t \triangleq u(W_{t \wedge \tau_D}) + \int_0^{t \wedge \tau_D} g(W_s)ds, \mathcal{F}_t; 0 \le t < \infty\}$ is a uniformly integrable martinagle.

Proof. First let's invoke Ito's formula:

$$\begin{aligned} u(W_t) &= u(W_0) + \int_0^t \sum_{i=1}^d \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} + \frac{1}{2}\int_0^t \Delta u(W_s)ds \\ &= u(W_0) + \int_0^t \sum_{i=1}^d \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} - \int_0^t g(W_s)ds \quad \mathbb{P}^x\text{-x a.s. for all } x \in D \end{aligned}$$

Thereofre,

$$u(W_{t \wedge \tau_D}) + \int_0^{t \wedge \tau_D} g(W_s)ds = u(W_0) + \int_0^{t \wedge \tau_D} \sum_{i=1}^d \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} \quad \mathbb{P}^x \text{ a.s..}$$

which is a martinagle. Now consider its quadratic variation:

$$\begin{aligned} < u(W_{t \wedge \tau_D}) + \int_0^{t \wedge \tau_D} g(W_s)ds > &= < \int_0^{t \wedge \tau_D} \sum_{i=1}^d \frac{\partial u}{\partial x_i}(W_s)dW_s^{(i)} > \\ &= \int_0^{t \wedge \tau_D} \sum_{i=1}^d \left(\frac{\partial u}{\partial x_i}(W_s) \right)^2 ds \end{aligned}$$

and by the fact that $\|u\|_{L^\infty(\overline{D})} < \infty$ we see it is uniformly integrable by **Problem1.5.24**. Then the convergence theorem of martinagles shows that this *Poisson Problem* as the desired representation as above.

For the second part of the problem, let $-g = -1$ and $f(x) = \frac{r-\|x\|}{2}$ on ∂B_r, then $u(x) = \frac{r^2-\|x\|^2}{d}$ is the analytic solution to the *Poisson Problem*. Therefore, by the representation we have established so far, we have the following equality:

$$\frac{r^2 - \|x\|}{d} = \mathbb{E}^x\left(\frac{r^2 - \|W_{\tau_{B_r}}\|^2}{d} + \tau_{B_r}\right) = \mathbb{E}^x\left(\tau_{B_r}\right)$$

□

Problem (4.2.26). *Suppose we remove condition (2.14) in* ***Proposition 2.7.*** *Show that $v(x) \triangleq \mathbb{P}^x\left(\tau_D = \infty\right)$ is harmonic in D, and if $a \in \partial D$ is regular, then $\lim_{x\to a; x\in D} v(x) = 0$. In particular, if every point of ∂D is regular, then with $v(x) = \mathbb{E}^x\left[f(W_{\tau_D} 1_{\tau_D<\infty})\right]$, the function $u + \lambda v$ is a bounded solution to the Dirichelet problem (D, f) for any $\lambda \in \mathbb{R}$. (It is possible to show that every bounded solution to (D, f) is of this form; see Port & Stone (1978), Theorem 4.2.12)./*

Proof. Intuitively we have

$$\mathbb{P}^y\left(\tau_D = \infty\right) = \mathbb{P}^0\left(\tau_D - \tau_{B_r} = \infty | W_{\tau_{B_r}} = y\right) = \mathbb{P}^x\left[\tau_D = \infty | W_{\tau_{B_r}} = y\right]$$

Therefore,

$$\begin{aligned}
\int_{\partial(x+B_r)} \mathbb{P}^y\left[\tau_D = \infty\right] d\mu(y) &= \int_{\partial[x+B_r]} \mathbb{P}^x\left[\tau_D - \tau_{x+B_r} = \infty | W_{\tau_{x+B_r}} = y\right] d\mu(y) \\
&= \int_{\partial[x+B_r]} \mathbb{P}^x\left[\tau_D - \tau_{x+B_r} = \infty\right] d\mu(y) \quad \text{by independence} \\
&= \mathbb{P}^x[\tau_D - \tau_{x+B_r} = \infty] \\
&= \mathbb{P}^x[\tau_D = \infty] \quad \text{since } \tau_{B_r+x} < \infty
\end{aligned}$$

Now suppose $a \in \partial D$ is regular, then from **Theorem 4.2.12 (iii)** we know that

$$\lim_{x\to a; x\in D} \mathbb{P}^x\left(\tau_D = \infty\right) \le \lim_{x\to a; x\in D} \mathbb{P}^x\left(\tau_D > \epsilon\right) = 0 \quad \forall \epsilon > 0$$

Now suppose every point on the boundary is regular, then v would be harmonic in D and continuous on $\overline{D}$. So consider the other part

$$\begin{aligned}
u \triangleq \mathbb{E}^x\left[f\left(W_{\tau_D}\right) 1_{\tau_D<\infty}\right] &= \mathbb{E}^x\left[\mathbb{E}^x\left[f\left(W_{\tau_D}\right) 1_{\tau_D<\infty} | \mathcal{F}_{\tau_{x+B_r}}\right]\right] \\
&= \mathbb{E}^x\left[u(W_{\tau_{x+B_r}})\right] \\
&= \int_{\partial(x+B_r)} u(x+y) d\mu(y)
\end{aligned}$$

hence u is a solution to (D, f), so $u + \lambda v$ is the solution to (D, f) since v has boudnary value zero. □

Excercise (4.2.27). *Let D be bounded with every boundary pont regular. Prove that every boundary point has a barrier.*

Proof. Let $a \in \partial D$, and let $f : \partial D \to \mathbb{R}$ be such that $f > 0$ on $\partial D \backslash \{a\}$ and $f(a) = 0$ be a continous bounded measurable function (since ∂D would be compact here). Then the function defined by (2.12) would be harmonic in D, namely,

$$u(x) \triangleq \mathbb{E}^x\left(f(W_{\tau_D})\right) \quad W \text{ is a standard one dim BM}$$

Now, by **Theorem 2.12** we see that u is continous on $\overline{D}$. So the only thing left to show is that u is positive on $\overline{D}$. First we see that u is not constant since it has to agree with f on the boudnary. Also, notice that $-u$ is also harmonic on D. So let's only look at the connected component of D such that a is in its boundary, call this region $\hat{D}$. Then by *maximum principle* on $-u$ we see that $-u(x) < 0$ for all $x\imath\hat{D}$. So we are done. □

Excercise (4.2.28). *A comoplex valued Brownian Motion is defined to be a process $\overline{W} = \{W_t^{(1)} + iW_t^{(2)}, \mathcal{F}_t; 0 \le t < \infty\}$, where $W = \{W_t^{(1)}, W_t^{(2)}, \mathcal{F}_t, 0 \le t, \infty\}$ is a two dim Brownian motion and $i = \sqrt{-1}$:*

- *Use* ***Theorem 3.4.13*** *to show that if $\overline{W}$ is a complex valued Brownian Motion and $f : \mathbb{C} \to \mathbb{C}$ is analytic and nonconstant, then under an appropriate condition, $f(\overline{W})$ is a complex valued Brownian Motion with a random time-change.*

- *With $\xi \in \mathbb{C}\backslash\{0\}$, show that $M_t \triangleq \xi e^{\overline{W}}, 0 \le t < \infty$ is a time changed, complex valued Brownian motion. (Hint: use **Problem 3.6.30**).*

Proof. Fot the first part, I am sensing Cauchy Riemann Formula, so let's use Ito's Lemma first to u, v where $f(x,y) = u(x,y) + iv(x,y)$. We know from Cauchy Riemann that u, v must be harmonic, therefore, we have

$$w(W_t) = w(0) + \sum_{i=1}^{2} \int_0^t w(W_s) dW_s^{(i)}$$

for $w = u, v$, therefore we see that

$$\begin{aligned} f(W_t) &= f(0) + \sum_{i=1}^{2} \int_0^t u_{x_i}(W_s) dW_s^{(i)} + \int_0^t iv_{x_i}(W_s) dW_s^{(i)} \\ &= f(0) + \sum_{i=1}^{2} \int_0^t f_{x_i}(W_s) dW_s^{(i)} \end{aligned}$$

so $f(W_t)$ is a local martiangle, so is $f(\overline{W}_t)$ and the independence of the 1-dim Brownian Motions makes $f(\overline{W}_t)$ satisfies the conditions for **Thoerem 3.4.13**.

For the second part, never read section 3.6, so not doing it. □

4.3 The One-Dimensional Heat Equation

4.3.B. Nonnegative Solutions of the Heat Equation

Excercise (4.3.8)**.** *(Widder's Uniqueness Theorem)*

1. *Let $u(t,x)$ be a nonnegative function of class $C^{1,2}$ defined on the strip $(0,T) \times \mathbb{R}$, where $0 < T \le \infty$, and assume that u satisfies (3.1) (the Heat Equation) on this strip and*

$$\lim_{t \downarrow 0; y \to x} u(t,y) = 0; \quad x \in \mathbb{R}.$$

Show that $u = 0$ on $(0,T) \times \mathbb{R}$. (Hint: Establish the uniform integrability of the martinagle $u(t-s, W_s); 0 \le s < t$.)

2. *Let u be as in (1), except now assume that $\lim_{t\downarrow 0, y \to x} u(t,y) = f(x); x \in \mathbb{R}$. Show that*

$$u(t,x) = \int_{\mathbb{R}} p(t;x,y) f(y) dy; \quad 0 < t < T, x \in \mathbb{R}.$$

Proof. (1) From **Corollary 3.7** we see that $\{u(t-s, W_s); \mathcal{F}_s, 0 \le s < t\}$ is a martinagle on $(\Omega, \mathcal{F}, \mathbb{P}^x)$ for all $x \in \mathbb{R}$, and $u(t,x) = \mathbb{E}^x [u(t-s, W_s)]$ for all $0 \le s < t < T, x \in \mathbb{R}$. The Ito's lemma gives us

$$u(t-s, W_s) = u(t, W_0) + \int_0^s \frac{\partial u}{\partial x}(t-\sigma, W_\sigma) dW_\sigma; \quad \mathbb{P}^x\text{-a.s.. } \forall x \in \mathbb{R}$$

lBy assumption we know that $\lim_{s \to t} u(t-s, W_t)(\omega) = 0$ pointwise a.s. by continuity of W_t.

Honestly, I have no idea how to show it is uniformly integrable.However, we can decompose the measure induced by F into the sum of two measures, one is absolutely continuous with respect to Lebesgue's measure, and another one is point measure, and use the properties of mollifier, uniformly convergence. So know that $dF = dF' + \sum_{n\in\mathbb{N}} \delta_{x_n}$, where $dF'(x) = f(x)dx$ where dx represents Lebesgue measure. So we have

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) dF = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) f(y) dy + \sum_{n\in\mathbb{N}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-x_n)^2}{2t}\right)$$

we send $t \to 0$ and note that $p(t;x,y)$ converges to zero uniformly outside of any ball centered at x when $x \neq y$, so the second part goes to zero. Note note that the first part is convolution between the mollifier and the measurable function f, this convolution converges to f, hence $f = 0$ by assumption. Therefore we have the result. The second part can be proven by the same method, but really, no idea how to prove it probabilistically. □

Before we continue, let's take a detour to *Brownian Motion with absorption at zero.*

Problem (2.8.6). *] Derive the transition density for Brownian Motion absorbed at the origin* $\{W_{t\wedge T_0}; \mathcal{F}_t, 0 \le t < \infty\}$*, by verifying that*

$$\mathbb{P}^x[W_t \in dy, T_0 > t] = p_-(t;x,y)\,dy \triangleq [p(t;x,y) - p(t;x,-y)]\,dy; \quad s > 0, t \ge 0, x, y > 0$$

Proof.

$$\begin{aligned}\mathbb{P}^x[W_t \in dy, T_0 > t] &= \mathbb{P}^x[W_t \in dy] - \mathbb{P}^x[W_t \in dy, T_0 \le t]\\ &= \mathbb{P}^x[W_t \in dy] - \mathbb{P}^x[W_t \in d(-y)]\end{aligned}$$

pluge in the transition probability of Brownian Motion then we are done, where the last equality is given by the *Reflection Principle* □

4.4. The Formulas of Feynman and Kac

4.4.A The Multidimensional Formula

Excercise (4.4.6). *Consider the Cauchy problem for the "quasilinear" parabolic equation*

$$\frac{\partial V}{\partial t} = \frac{1}{2}\Delta V - \frac{1}{2}\|\triangle V\|^2 + k; \quad in\ (0,\infty) \times \mathbb{R}^d$$

$$V(0,x) = 0; \quad x \in \mathbb{R}^d$$

(linear in $(\frac{\partial V}{\partial t})$ *and the Laplacian* ΔV*, nonlinear in the gradient* $\triangle V$*), where* $k : \mathbb{R}^d \to [0,\infty)$ *is a continuous function. Show that the only solution* $V : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ *which is continuous on its domain, of class* $C^{1,2}$ *on* $(0,\infty) \times \mathbb{R}^d$*, and satisfies the quadratic growth condition for every* $T > 0$*:*

$$-V(t,x) \le C + a\|x\|^2; \quad (t,x) \in [0,T] \times \mathbb{R}^d$$

where $T > 0$ *is arbitrary and* $0 < a < \frac{1}{2}Td$*, is given by*

$$V(t,x) = -\log \mathbb{E}^x\left[\exp\left\{-\int_0^t k(W_s)ds\right\}\right]$$

Proof. Let $t \le T$ and denote $A_s = exp\left\{-\int_0^s k(W_r)dr\right\}$ and $B_s = exp\left\{-u(t-s, W_s)\right\}$, note that B_s has finite total variation, so $< B >_t = 0$, hence $< A, B >_t = 0$. So by the stochastic version of integration by parts formula we have

$$dA_sB_s = A_s dB_s + B_s dA_s$$

where we know that $dB_s = -exp\left\{-\int_0^s k(W_r)dr\right\}k(W_s)$ since the integral is defined in Lebesgue's sense. Now let's use Ito's lemma on dA_s.:

$$\begin{aligned}dA_s &= \frac{\partial u}{\partial t}(t-s, W_s)A_s ds + A_s \sum_{i=1}^d \frac{\partial u}{\partial x_i}(t-s, W_s)dW_s^{(i)} + \left(\frac{1}{2}\|\triangle u(t-s, W_s)\| - \frac{1}{2}\Delta u(t-s, W_s)\right) A_s ds\\ &= A_s k(W_s)ds + A_s \sum_{i=1}^d \frac{\partial u}{\partial x_i}(t-s, W_s)dW_s^{(i)}\end{aligned}$$

So we have

$$\begin{aligned}d(A_sB_s) &= A_sB_s k(W_s)ds + A_sB_s \sum_{i=1}^d \frac{\partial u}{\partial x_i}(t-s, W_s)dW_s^{(i)} - A_sB_s k(W_s)ds\\ &= A_s \sum_{i=1}^d \frac{\partial u}{\partial x_i}(t-s, W_s)dW_s^{(i)}\end{aligned}$$

Now let $R_n = \inf_{t \ge 0}\{\|W_t\| \ge n\}$ and integrate this from 0 to $t \wedge R_n$ and take expectation with repsect to $\mathbb{P}^x$ for any fixed $x \in \mathbb{R}^n$ to get

$$\begin{aligned}u(t,x) &= \mathbb{E}^x\left[\exp\left\{-u(t - t\wedge R_n, W_{t\wedge R_n}) - \int_0^{R_n \wedge t} k(W_s)ds\right\}\right]\\ &= \mathbb{E}^x\left[\exp\left\{-\int_0^t k(W_s)ds\right\}1_{R_n > t}\right] + \mathbb{E}^x\left[\exp\left\{-u(t - R_n, W_{R_n}) - \int_0^{R_n} k(W_s)ds\right\}1_{R_n \le t}\right]\end{aligned}$$

Now the first term converges to $\mathbb{E}^x\left[\exp\left\{-\int_0^t k(W_s)ds\right\}\right]$ by bounded convergence theorem since $k \geq 0$. Now consider the integrand of the second term: for some $C > 0$ we have

$$
\begin{aligned}
\mathbb{E}^x\left[\exp\left\{-u(t-R_n, W_{R_n}) - \int_0^{R_n} k(W_s)ds\right\} 1_{R_n \leq t}\right] &\leq Ce^{an^2}\mathbb{P}^x\{R_n \leq t\} \\
&\leq Ce^{an^2}\left(\mathbb{P}^x[T_n \leq t] + \mathbb{P}^x[T_{-n} \leq t]\right)^d \\
&= Ce^{an^2}\left(\mathbb{P}^0[T_{n-x} \leq t] + \mathbb{P}^0[T_{-n-x} \leq t]\right)^d \\
&= C'e^{an^2}\left(\int_{|n+x|t^{-1/2}}^{\infty} e^{-y^2/2}dy + \int_{|-n-x|t^{-1/2}}^{\infty} e^{-y^2/2}dy\right)^d
\end{aligned}
$$

Let's denote $n_x = \min\{|n-x|, |-n-x|\}$, then above is less than the following

$$
C'e^{an^2} 2\frac{1}{n_x} e^{-\frac{n_x^2}{2t}}
$$

by the condition imposed on a, we see it goes to zero. $\square$

Excercise (4.4.7)**.** *Let ψ be the solution to*

$$
(\alpha + k)\psi = \frac{1}{2}\Delta\psi + f; \quad \text{on } \mathbb{R}^d
$$

and let $f : \mathbb{R}^k \to \mathbb{R}$ and $k : \mathbb{R}^k \to \mathbb{R}^+$ be continous, with

$$
\mathbb{E}^x\left[\int_0^\infty |f(W_t)| \exp\left\{-\alpha t - \int_0^t k(W_s)ds\right\} dt < \infty\right]; \quad \forall x \in \mathbb{R}^d,
$$

for some $\alpha > 0$, the same α as in the definition of ψ. Now let's define z to be

$$
z(x) = \mathbb{E}^x\left[\int_0^\infty f(W_t) \exp\left\{-\alpha t - \int_0^t k(W_s)ds\right\} dt\right].
$$

If ψ is bounded, show that $\psi = z$; if ψ is nonnegative, then $\psi \geq z$. (Hint: Use Problem 2.25).

Proof. Let's first try the usual method for proving uniqueness: suppose ψ is a bounded solution, we show that it must of of the form of z, then we should be done here. Let $A_t = \int \exp\left\{-\alpha t - \int_0^t k(W_s)ds\right\}$ and consider (reason same as the previous problem):

$$
d\left[\psi(W_t)A_t\right] = A_t d\psi(W_t) + \psi(W_t)dA_t
$$

where $dA_t = -A_t(\alpha + k)$ and

$$
\begin{aligned}
d\psi(W_t) &= \sum_{i=1}^d \frac{\partial\psi}{\partial x_i}(W_t)dW_t^{(i)} + \frac{1}{2}\Delta\psi(W_t)dt \\
&= \sum_{i=1}^d \frac{\partial\psi}{\partial x_i}(W_t)dW_t^{(i)} + (\alpha + k(W_t))\psi(W_t) - f
\end{aligned}
$$

combine them to get

$$
\begin{aligned}
d\left[\psi(W_t)A_t\right] &= A_t d\psi(W_t) + \psi(W_t)dA_t \\
&= A_t \sum_{i=1}^d \frac{\partial\psi}{\partial x_i}(W_t)dW_t^{(i)} - A_t f
\end{aligned}
$$

Now integrate from 0 to infinity, we can do this because we have the integrable condition. By the assumption that ψ is (uniformly) bounded, taking expectation with respect to $\mathbb{P}^x$, we get the desired result. $\square$

Chapter 5 Stochastic Differential Equations

5.2. Strong Solutions

Problem (5.2.7). *Suppose $g(t)$ is continuous and satisfies*

$$0 \le g(t) \le \alpha(t) + \beta \int_0^t g(s)ds; \quad 0 \le t \le T$$

with $\beta \ge 0$ and $\alpha : [0,T] \to \mathbb{R}$ integrable. Then

$$g(t) \le \alpha(t) + \beta \int_0^t \alpha(s) e^{\beta(t-s)} ds; \quad 0 \le t \le T0$$

Proof. Denote $G(t) \triangleq \int_0^t g(s)ds$, then the above inequality reads

$$\begin{aligned} &G'(t) \le \alpha(t) + \beta G(s) \\ \Rightarrow &\frac{d}{dt} e^{-\beta t} G(t) = e^{-\beta t} G'(t) - \beta e^{-\beta t} G(t) \le e^{-\beta t} \alpha(t) \end{aligned}$$

Now integrate on both sides from 0 to t with dummy variable s we have

$$e^{-\beta t} G(t) \le \int_0^t e^{-\beta s} \alpha(s) ds \Rightarrow G(t) = \int_0^t g(s)ds \le \int_0^t e^{\beta(t-s)} \alpha(s) ds$$

Put this back into the original inequality we shall get the desired result. □

Problem (5.2.10). *For every T, show that there existst $C > 0$ depending only on K and T such that for the interations:*

$$X_t^{(n+1)} = \xi + \int_0^t b(s, X_s^{(n)}) ds + \int_0^t \sigma\left(s, X_s^{(n)}\right) dW_s; 0 \le t < \infty$$

for all $n \ge 0$ where $X_t^{(0)} \equiv \xi$, where σ and b satisifes the global Lipschitz and linear growht conditions, namely:

$$\begin{aligned} \|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| &\le K\|x-y\|, \\ \|b(t,x)\|^2 + \|\sigma(t,x)\|^2 &\le K^2\left(1 + \|x\|^2\right) \end{aligned}$$

with initial condition ξ being square integrable. Show that we have

$$\mathbb{E}[\|X_t^{(k)}\|^2] \le C\left(1 + \mathbb{E}\|\xi\|^2\right) e^{Ct}$$

Proof.

$$\begin{aligned} \mathbb{E}[\|X_t^{(k)}\|^2] &\le C\left(\mathbb{E}\|\xi\|^2 + \mathbb{E}\int_0^t \|b(s, X_s^{(k-1)})\|^2 ds + \sum_{i=1}^d \mathbb{E}\left[\sum_{j=1}^d \int_0^t \sigma^{(i,j)}\left(s, X_s^{(k-1)}\right) dW_s^{(j)}\right]^2\right) \\ &\le C\left(\mathbb{E}\|\xi\|^2 + \mathbb{E}\int_0^t \|b(s, X_s^{(k-1)})\|^2 ds + \sum_{i=1}^d \mathbb{E}\left[\sum_{j=1}^d \int_0^t \sigma^{(i,j)}\left(s, X_s^{(k-1)}\right)^2 dW_s^{(j)}\right]\right) \\ &= C\left(\mathbb{E}\|\xi\|^2 + \mathbb{E}\int_0^t \|b(s, X_s^{(k-1)})\|^2 ds + \int_0^t \|\sigma(s, X_s^{(k-1)})\|^2 ds\right) \\ &\le C\left(\mathbb{E}\left[\|\xi\|^2 + K^2 \int_0^t 1 + \|X_s^{(k-1)}\|^2 ds\right]\right) \\ &= C\mathbb{E}\|\xi\|^2 + CK^2 t + CK^2 \int_0^t \mathbb{E}\|X_s^{(k-1)}\| ds \end{aligned}$$

Note that here C is independent of k. Note that $t \le T$, so we can absorbe t and K into C, the simplified inequality reads

$$\mathbb{E}[\|X_t^{(k)}\|^2] \le C(1 + \mathbb{E}\|\xi\|^2) + C\int_0^T \mathbb{E}[\|X_s^{(k-1)}\|^2 ds$$

Iterating this inequality gives us the desired result, where the Taylor expansion would kick in. □

Problem (5.2.11). *Show that the process constructed in the proof of **Theorem 5.2.10** satisfies requirement:*

$$X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

Proof. In the proof, X_t is constructed as a pointwise sup limit of $X^{(k)}$ on any finite interval $[0, T]$, that is,

$$\sup_{t\in[0,T]} \|X_t(\omega) - X_t^{(k)}(\omega)\| \to 0; \quad \forall T > 0.$$

From this condition we see that

$$\begin{aligned}\int_0^t \|b(s, X_s) - b(s, X_s^{(k)})\|ds &\leq \int_0^t K^2\|X_s - X_s^{(k)}\|ds \\ &\leq TK^2 \sup_{t\in[0,T]} \|X_t - X_t^{(k)}\|\end{aligned}$$

which converges to zero a.s.. By simimar arugument we see that

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}\left[\|\int_0^t \sigma(s, X_s) - \sigma(s, X_s^{(k)})ds\|^2\right] &\leq \lim_{k\to\infty} \mathbb{E}\left[\int_0^t \|\sigma(s, X_s) - \sigma(s, X_s^{(k)})\|^2 ds\right] \\ &\leq \lim_{k\to\infty} \mathbb{E}\left[\int_0^t K^2\|X_s - X_s^{(k)}\|^2 ds\right]\end{aligned}$$

By (2.15) and (2.17), we can use Dominated Convergence theorem to move the limit inside the integral and see it goes to zero since we have uniformly convergence for each ω. So the second term converges to the corresponding "X term" in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, hence converges in probability.

So far we have $X^{(k)}$ converges to X a.s. and $X^{(k)}$ converges to $\xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$ in probability, so those two things are equal a.s.. □

Excercise (5.2.17). *The stochastic equation*

$$X_t = 3\int_0^t X_s^{\frac{1}{3}} ds + 3\int_0^t X_s^{\frac{2}{3}} dW_s$$

has uncountably many strong solutions of the form

$$X_t^\theta = \begin{cases} 0; & 0 \leq t < \beta_\theta \\ W_t^3; & \beta_\theta \leq t < \infty \end{cases}$$

where $0 \leq \theta \leq \infty$ *and* $\beta \triangleq \inf\{s \geq \theta; W_s = 0\}$.

Proof. In another words, $X_t^\theta = 1_{t\geq\beta_\theta} W_t^3 = W_t^3 - W_{t\wedge\beta_\theta}^3$. Use Ito's formula we have

$$\begin{aligned}W_t^3 &= 3\int_0^t W_s^2 dW_s + 3\int_0^t W_s ds \\ W_{t\wedge\beta_\theta}^3 &= 3\int_0^t W_{s\wedge\beta_\theta}^2 dW_s + 3\int_0^t W_{s\wedge\beta_\theta} ds\end{aligned}$$

so

$$X_t = 3\int_0^t W_s^2 - W_{s\wedge\beta_\theta}^2 dW_s + 3\int_0^t W_s - W_{s\wedge\beta_\theta} ds$$

Observe that

$$W_s - W_{s\wedge\beta_\theta} = \begin{cases} 0 & \text{when } t < \beta_\theta \\ W_s & \text{when } \beta_\theta \leq t < \infty \end{cases} = X_s^{\frac{1}{3}}$$

and

$$W_s^2 - W_{s\wedge\beta_\theta}^2 = \begin{cases} 0 & \text{when } t < \beta_\theta \\ W_s^2 & \text{when } \beta_\theta \leq t < \infty \end{cases} = X_s^{\frac{2}{3}}$$

so we have the desired result for $\theta \in \mathbb{R}^+$. When $\theta = 0$, then $X_t = W_t^3$ which solves the equation, and when $\theta = \infty$, then $X_t \equiv 0$ is the trivial solution. □

Excercise (5.2.19). *Suppose that in* ***Proposition 2.18*** *we drop condition (v) but strengthen condition (iv) to*

$$b_1(t,x) < b_2(t,x); \quad 0 \le t < \infty, x \in \mathbb{R}$$

Then the conclusion (2.32) still holds. Hint: For each integer $m \ge 3$, construct a Lipschitz-continuous function $b_m(t,x)$ such that $b_1(t,x) \le b_m(t,x) \le b_2(t,x)$.

Proof. Let's say $\mathbb{E}\int_0^t |\sigma(s,X_s)|^2 ds < \infty$. Define

$$\Delta_t \triangleq X^{(1)} - X^{(2)} = \int_0^t \{b(s,X_s^{(1)}) - b(s,X_s^{(2)})\}ds + \int_0^t \{\sigma(s,X_s^{(1)}) - \sigma(s,X_s^{(2)})\}dW_s$$

Note that we have the condition

$$|\sigma(t,x) - \sigma(t,y)| \le h|x-y| \quad \forall x \in \mathbb{R}^+; x,y \in \mathbb{R}$$

where $h : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $h(0) = 0$o and

$$\int_{(0,\epsilon)} h^{-2}(x)dx = \infty, \quad \forall \epsilon > 0$$

That means there exists a positive and strictly decreasing sequence $\{\alpha_n\}$ such that $\alpha_n \downarrow 0$ such that $\int_{\alpha_{n+1}}^{\alpha_n} h^{-2}(x)dx = n$ for all $n \ge 1$. Now by Urysoln or variation of it, there exists a function ρ_n for each n such that $0 \le \rho_n(x) \le \frac{2}{nh(x)}$ wiht $\int_{\alpha_{n+1}}^{\alpha_n} \rho_n(x)dx = 1$ and $\rho_n(x)$'s support is contained in (α_{n+1}, α_n).

We now define the following function

$$\varphi_n(x) = 1_{[0,\infty)}(x) \int_0^x \int_0^y \rho_n(u)dudy$$

so φ is continuously twice differentiable and $|\varphi'| < 1$ by the property of ρ_n. Also, $\lim_{n\to\infty} \varphi_n(x) = x$. Futhermore, the sequence $\{\varphi_n\}$ is nondecreasing.

Now, let $f(t,x) \triangleq \frac{1}{2}(b_1(t,x) + b_2(t,x))$, and η_ϵ be a set of mollifiers (in usual sense) and define $f_\epsilon \triangleq f * \eta_\epsilon$. So f_ϵ is also smooth hence Lipschitz-continous in a compact subset of $\mathbb{R}^+ \times \mathbb{R}$. For each $m \ge 3$, let ϵ_n be so small such that f_{ϵ_m} is between b_1 and b_2 on the set $0 \le t \le m$ and $|x| \le m$ (we can do this because of the uniformly convergence) and let $b_n \triangleq f_{\epsilon_n}$. Now let $R_m^{(i)} = \inf_{t\ge 0}\{|X_t^{(i)}| = m\}$ and let $R_m = R_m^{(1)} \wedge R_m^{(2)}$.

Now let τ be any positive number and $t = \tau \wedge R_m$, then by the same argument, we can have the relation

$$\begin{aligned}
\mathbb{E}\varphi_n(\Delta_t) - \frac{t}{n} &\le \mathbb{E}\int_0^t \varphi_n'(\Delta_s)\left[b_1(s,X_s^{(1)} - b_1(s,X_s^{(2)})\right] \\
&= \mathbb{E}\left[\int_0^t \varphi_n'(\Delta_s)[b_1(t,X_s^{(1)}) - b_m(t,X_s^{(1)})]ds\right] \\
&+\mathbb{E}\left[\int_0^t \varphi_n'(\Delta_s)[b_m(t,X_s^{(1)}) - b_m(t,X_s^{(2)})]ds\right] \\
&+\mathbb{E}\left[\int_0^t \varphi_n'(\Delta_s)[b_m(t,X_s^{(2)}) - b_2(t,X_s^{(2)})]ds\right] \\
&\le \mathbb{E}\left[\int_0^t \varphi_n'(\Delta_s)[b_m(t,X_s^{(1)}) - b_m(t,X_s^{(2)})]ds\right] \\
&\le K_m \int_0^t \mathbb{E}[\Delta_s^+]ds
\end{aligned}$$

Let $n \to \infty$ and use Gronwall inequality we see that $\mathbb{E}[\Delta_t^+] = 0$. Now, since t depends on m as well, we vary m and use t_m instead to indicate this dependency. As $m \to \infty$, by DCT we have desired result. □

Excercise (5.2.20). *Suppose that the coefficients $\sigma : \mathbb{R} \to (0,\infty)$ and $b : \mathbb{R} \to \mathbb{R}$ are of class C^2, C^1 respectively; that $b' - \frac{1}{2}\sigma\sigma'' - \frac{b\sigma'}{\sigma}$ is bounded; and that $\frac{1}{\sigma}$ is not integrable at either $\pm\infty$. Then*

$$X_t = \xi + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s; \quad 0 \le t < \infty$$

has a unique, strong solution X. (Hint: Consider the function $f(x) = \int_0^x \frac{du}{\sigma(u)}$ and apply Ito's rule to $f(X_t)$.)

Proof. I guess we are also assuming that f given by the hint is defined when $x \neq \pm\infty$, Picard's Iteration method might not work, so try the hint first:

$$\begin{aligned} f(X_t) &= f(\xi) + \int_0^t \frac{b(X_s)}{\sigma(X_s)} ds + W_t - \frac{1}{2}\int_0^t \sigma'(X_s) ds \\ &= f(\xi) + \int_0^t \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2}\sigma(X_s) ds + W_t \end{aligned}$$

Note that $\frac{d}{dx}\left(\frac{b(x)}{\sigma(x)} - \frac{1}{2}\sigma(x)\right) = \frac{1}{\sigma(x)}\left(b'(x) - \frac{1}{2}\sigma''(x) - \frac{\sigma'(x)b(x)}{\sigma(x)}\right)$. We need some boundedness condition of $\frac{1}{\sigma}$. □

Excercise (5.2.27). *Solve explicitly the one-dimensional equation*

$$dX_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right) dt + \sqrt{1+X_t^2} dW_t$$

Proof. Let $\sigma(x) = \sqrt{1+x^2}$ and $b(x) = \sigma(x) + \frac{1}{2}x$. We see that

$$\frac{b(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x) = 0$$

Using the notation from **Excercise 5.2.20** we get

$$df(X_t) = dW_t$$

and suppose $X_0 = 0$ a.s., then $f(X_t) = \int_0^{X_t} \frac{du}{\sigma(u)} = \ln\left(X_t + \sqrt{1+X_t^2}\right) = W_t$. Solve to get

$$X_t = \frac{1}{2}e^{-W_t}(e^{2W_t} - 1)$$

□

Excercise (5.2.28).

1. *Suppose that there exists an $\mathbb{R}^d$-valued function $u(t,y) = (u_i(t,y))_{1\le i\le d}$ of class $C^{1,2}([0,\infty)\times\mathbb{R}^d)$, such that*

$$\frac{\partial u_i}{\partial t}(t,y) = b_i(t,u(t,y)), \quad \frac{\partial u_i}{\partial x_j}(t,y) = \sigma_{i,j}(t,u(t,y)); \quad 1 \le i,j \le d$$

   hold on $[0,\infty)\times\mathbb{R}^d$, where each $b_i(t,x)$ is continuous and each $\sigma_{i,j}$ is of class $C^{1,2}$ on $[0,\infty)\times\mathbb{R}^d$. Show then that the process

$$X_t \triangleq u(t,W_t); \quad 0 \le t < \infty$$

   where W is a d-dimensional Brownian motion, sovles the Fisk-Stratonovich equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)\circ dW_t$$

2. *Use the above result to find the unique, strong solution of the one-dimensional Ito equation*

$$dX_t = \left[\frac{2}{1+t}X_t - a(1+t)^2\right] dt + a(1+t)^2 dW_t; \quad 0 \le t < \infty$$

Proof. Recall the *Fisk-Stratonovich Integral* is defined for semi-Mtgs X, Y as

$$\int_0^t Y_s \circ dX_s \triangleq \int_0^t Y_s M_s + \int_0^t Y_s dB_s + \frac{1}{2} < N, M >_t; 0 \le t < \infty$$

where the meaning of each term is apperant. So turn the SDE in Ito's sense, we need to solve

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t + d < \sigma(\cdot, X_\cdot), W_\cdot >_t$$

Now we use Ito's Formula on $u(t,W_t)$:

$$\begin{aligned} du(t,W_t) &= \frac{\partial u}{\partial t}(t,W_t)dt + \sum_{i=1}^d \frac{\partial u}{\partial x_i}(t,W_t)dW_t + \frac{1}{2}\sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}(t,W_t)dt \\ &= b(t,u(t,W_t))dt + \sigma(t,u(t,W_t))dW_t + \frac{1}{2}\sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}(t,W_t)dt \end{aligned}$$

The definition of $\sigma(t,X_t)\circ dW_t$ is unclear since σ is a matrix so the quadratic variation makes no sense. Howevere if the quadratic variation is defined to be $\sum_{j=1}^d \sigma_{i,j}(t,X_t)dW_t^j$, then we are done here. □

5.3. Weak Solutions

5.3.B. Weak Solutions by Means of the Girsanov Theorem

Problem (5.3.13). *Consider the stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

with $\sigma(t,x)$ be a $d \times d$ nonsingular matrix for all $t \geq 0, x \in \mathbb{R}^d$. Assume that $b(t,x)$ is uniformly bounded, the smallest eigenvalue of $\sigma(t,x)\sigma^{tr}(t,x)$ is uniformly bounded away from zero, and the equation

$$dX_t = \sigma(t, X_t)dW_t; \quad 0 \leq t \leq T$$

has a weak solution with initial distribution μ. Show that the first equation also has a weak solution for $0 \leq t \leq T$ with initial distribution μ.

Proof. Let's say that the problem meant $\sigma(t,x)$ has smallest eigenvalue uniformly bounded away from zero, since it is this case on the PDF and it makes the problem easier. In this case, $\sigma(t,x)$ is invertible and the eigenvalues of $\sigma^{-1}(t,x)$ is uniformly bounded, hence $\sigma^{-1}(t,x)b(t,x)$ is also uniformly bounded since b is uniformly bounded.

Let $Z_t = \exp\left\{\sum_{i=1}^d \int_0^t \sum_{j=1}^d \sigma_{i,j}^{-1}(s, X_s)b^j(s, X_s)dW_s - \frac{1}{2}\int_0^t \|\sigma^{-1}(s, X_s)b(s, X_s)\|^2 ds\right\}$, by **Corollary 3.5.13** we see that Z_t is a martinagle for $0 \leq t \leq T$ where W_t is the Brownian motion in the weak solution of the second SDE. Define $\widetilde{\mathbb{P}}$ by $\widetilde{\mathbb{P}}(A) = \mathbb{E}[1_A A_T]$ for all $A \in \mathcal{F}_T$, then we see that

$$\widetilde{W}_t = W_t - \int_0^t \sigma^{-1}(s, X_s)b(s, X_s)ds; 0 \leq t \leq T$$

is a Brownian motion under $(\Omega, \mathcal{F}_T, \widetilde{\mathbb{P}})$. Writing it as componentwise as the following

$$\widetilde{W}_t^{(i)} = W_t^{(i)} - \int_0^t \sum_{j=1}^d \sigma_{i,j}^{-1} b_j(s, X_s)ds; \quad 1 \leq i \leq d$$

is a standard one dimensional Brownian motion under the same probability space. Now denote $c^i(t, X_t) \triangleq \sum_{j=1}^d \sigma_{i,j}^{-1}(t, X_t)b_j(t, X_t)$.

By **Problem 3.5.6** we see that for all Y_s such that $\mathbb{P}[\int_0^t Y_s^2 ds < \infty] = 1$ for $0 \leq t \leq T$, then we have

$$Y_t d\widetilde{W}_t^i = Y_t dW_t^i - Y_t c^i(t, X_t)dt; \quad 0 \leq t \leq T, \quad a.s. \mathbb{P}, \widetilde{\mathbb{P}}$$

Assume WLOG that $\int_0^t \|\sigma_s\| ds < \infty$ a.s.. So in matrix form we have

$$\begin{aligned}\sigma(t, X_t)d\widetilde{W}_t &= \sigma(t, X_t)dW_t - b(t, X_t)dt \\ &= dX_t - b(t, X_t)dt; \quad \mathbb{P}, \widetilde{\mathbb{P}}\end{aligned}$$

Since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ agree on $\mathcal{F}_0$, then we are done here, and the solution is $(\Omega, \mathcal{F}_t, \widetilde{\mathbb{P}})$ and $X_t, \widetilde{W}_t$. ☐

Problem (5.3.15). *Suppose $b_i(t,y), \sigma_{ij}(t,y); 1 \leq i \leq d$ and $1 \leq j \leq r$ are progressively measurable functionals from $[0,\infty) \times C[0,\infty)^d$ into $\mathbb{R}$ satisfiying*

$$\|b(t,y)\|^2 + \|\sigma(t,y)\|^2 \leq K\left(1 + \max_{0 \leq s \leq t}\|y(s)\|^2\right); \quad \forall 0 \leq t < \infty, \quad y \in C[0,\infty)^d,$$

where K is a positive constant. If $(X, W), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ is a weak solution to

$$dX_t = b(t, X)dt + \sigma(t, X)dW_t$$

with $\mathbb{E}\|X_0\|^{2m} < \infty$ for some $m \geq 1$, show that for any finite $T \geq 0$, we have

$$\mathbb{E}[\max_{0 \leq s \leq t}\|X_s\|^{2m}] \leq C\left(1 + \mathbb{E}\|X_0\|^{2m}\right)e^{Ct}; \quad 0 \leq t \leq T$$

$$\mathbb{E}[\|X_t - X_s\|^{2m}] \leq C\left(1 + \mathbb{E}[\|X_0\|^{2m}]\right)(t-s)^m; \quad 0 \leq s < t \leq T$$

where C is a positive constant depending only on m, T, K, d.

Proof. Under corresponding probability space and Brownian Motion, we have

$$X_t = X_0 + \int_0^t b(s,X)ds + \int_0^t \sigma(s,X)dW_s$$

Now by **Problem 3.3.29** and **Remark 3.3.30** and Funini's

$$\mathbb{E}\|\max_{0\le s\le t}\int_0^s \sigma(u,X)dW_u\|^{2m} \le \Lambda_m \int_0^t \mathbb{E}[\|\sigma(s,X)\|^{2m}]ds \le \Lambda_m \int_0^t \mathbb{E}\left[1+\max_{0\le s\le t}\|X_s\|^2\right]^m$$

and

$$\mathbb{E}\left(\|\int_0^t b(s,X)ds\|^{2m}\right) \le \int_0^t \mathbb{E}\left(1+\max_{0\le s\le t}\|X_s\|^2\right)^m ds$$

$(a+b)^m = \left(2a\frac{1}{2}+\frac{1}{2}b\right)^m \le 2^{m-1}a^m + 2^{m-1}b^m$ by Jessen's. Let's denote $B_t \triangleq \mathbb{E}\left[\max_{0\le s\le t}\|X_s\|^{2m}\right]$, so we have

$$\begin{aligned} B_t &\le C\left(2\mathbb{E}[\|X_0\|^{2m}] + 2\Lambda_m Dt + 2\Lambda_m D\int_0^t B_s ds\right) \\ &\le C(2\mathbb{E}[\|X_0\|^{2m}]+1) + C\int_0^t B_s ds \end{aligned}$$

For some C, D where the C's in each line could be different. Then by Gronwall we are done.

Now for the next assersion, is a direct consequence of the first one with the fact that $e^{t-s} \ge t-s$ for $t > s$. □

5.4. The Martinagle Problem of Stroock and Varadhan

A. Some Fundamental Martingales

Problem (5.4.3). *Let $b_i(t,y), \sigma_{ij}(t,y) : [0,\infty)\times C[0,\infty)^d \to \mathbb{R}$ be progressively measurable functionals for all $1 \le i \le d, 1 \le j \le r$. We define the diffusion matrix $a(t,y)$ with components*

$$a_{i,k} \triangleq \sum_{j=1}^r \sigma_{ij}(t,y)\sigma_{kj}(t,y); \quad 0 \le t < \infty, y \in C[0,\infty)^d.$$

Suppose that $(X,W),(\Omega,\mathcal{F},\mathbb{P}),\{\mathcal{F}_t\}$ is a weak solution to the functional stochastic differential equation

$$dX_t = b(t,X)dt + \sigma(t,X)dW_t$$

and set that

$$(\mathcal{A}_t')(y) = \frac{1}{2}\sum_{i=1}^d\sum_{k=1}^d a_{ik}(t,y)\frac{\partial^2 u(y(t))}{\partial x_i \partial x_k} + \sum_{1\le i\le d} b_i(t,y)\frac{\partial u(y(t))}{\partial x_i};$$
$$0 \le t < \infty, u \in C^2(\mathbb{R}^d), y \in C[0,\infty)^d$$

Then show that for any function $f, g \in C[0,\infty)\times\mathbb{R}^d \cap C^{1,2}((0,\infty)\times\mathbb{R}^d)$, the process

$$M_t^f = \triangleq f(t,X_t) - f(0,X_0) - \int_0^t \left[\frac{\partial f}{\partial s} + \mathcal{A}_s' f\right](s,X)ds, \quad \mathcal{F}_t; 0 \le t < \infty$$

is in $\mathcal{M}^{c,loc}$, and

$$< M^f, M^g >_t = \sum_{1\le i,j\le d}\int_0^t a_{ik}(s,X)\frac{\partial f}{\partial x_i}(s,X_s)\frac{\partial g}{\partial x_k}(s,X_s)ds.$$

Furthermore, if $f \in C_0([0,\infty)\times\mathbb{R}^d)$ and for each $0 < T < \infty$ we have

$$\|\sigma(t,X)\| \le K_T; \quad 0 \le t \le T, \quad y \in C[0,\infty)^d,$$

Where K_T is a constant depending on T, then $f \in \mathcal{M}_2^c$.

Proof. By definition we have

$$X_t = X_0 + \int_0^t b(s,X)ds + \int_0^t \sigma(s,X)dW_s; 0 \le t < \infty; \quad a.s.$$
$$\int_0^t \{|b_i(s,X)| + \sigma_{i,j}^2(s,X)\}ds < \infty \quad a.s..$$

let $S_n \triangleq \inf_{t\ge 0}\{\|X_t\| \ge n \text{ or } \int_0^t \|\sigma(s,X)ds \ge n\|\}$. In this case, since b and σ are progressively measruable, then $X_{t\wedge S_n}$ is a Martinagle hence X_t is a semi-martingale. Now let's apply Ito's formula to $f(X_t)$:

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X)ds + \sum_{1\le i\le d}\int_0^t \frac{\partial f}{\partial x_i}(s,X_s)b_i(s,X)ds + \sum_{1\le i\le d}\int_0^t \frac{\partial f}{\partial x_i}(s,X_s)d(\sum_{j=1}^d \sigma_{ij}(s,X)dW_s^j)$$
$$+\frac{1}{2}\sum_{1\le i\le d}\sum_{1\le j\le d}\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)d < \sum_{1\le k\le d}\int_0^s \sigma_{i,k}(u,X)dW_s^k, \sum_{1\le k\le d}\int_0^s \sigma_{j,k}(u,X)dW_s^k >$$

simplify above to get

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X)ds + \sum_{1\le i\le d}\int_0^t \frac{\partial f}{x_i}(s,X_s)b_i(s,X)ds + \sum_{1\le i\le d}\sum_{1\le j\le d}\int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\sigma_{ij}(s,X)dW_s^j$$
$$+\frac{1}{2}\sum_{1\le i,j\le d}\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)\sum_{k=1}^d \sigma_{ik}(s,X)\sigma_{jk}(s,X)ds$$

So if we add the notation $a_{i,j}$ and $\mathcal{A}'_t$, the above equation becomes

$$f(t,X_t) - f(0,X_0) = \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \int_0^t (\mathcal{A}'_s f_s)(X_s)ds + \sum_{1\le i,j\le d}\int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\sigma_{ij}(s,X)dW^j$$

so $M_t^f = \sum_{1\le i,j\le d}\int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\sigma_{ij}(s,X)dW^j$ is a local Mtg with stopping time S_n.

For the second assersion,

$$< M^f, M^g >_t =< \sum_{1\le i,j\le d}\int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\sigma_{ij}(s,X)dW^j, \sum_{1\le i,j\le d}\int_0^t \frac{\partial g}{\partial x_i}(s,X_s)\sigma_{ij}(s,X)dW^j >$$
$$=< \sum_{1\le j\le d}\int_0^t \sum_{1\le i\le d}\frac{\partial f}{\partial x_i}(s,X)\sigma_{ij}(s,X)dW_j, \sum_{1\le j\le d}\int_0^t \sum_{1\le k\le d}\frac{\partial g}{\partial x_k}(s,X_s)\sigma_{kj}(s,X)dW_j >$$
$$= \sum_{1\le j\le d}\int_0^t \left(\sum_{1\le i\le d}\frac{\partial f}{\partial x_i}\sigma_{ij}(s,X)\right)\left(\sum_{1\le k\le d}\frac{\partial g}{\partial x_k}\sigma_{kj}(s,X)\right)ds$$
$$= \sum_{1\le i,j\le d}\int_0^t a_{ik}(s,X)\frac{\partial f}{\partial x_i}(s,X_s)\frac{\partial g}{\partial x_k}(s,X_s)ds$$

For the third assertion, if the norm of σ is bounded for $0 \le t \le T$, then the stochastic integral is defined by the boundedness of σ with either progressively measurable or the absolutely continuous of the quadratic variation of the Brownian motion, so we do need the stopping time anymore. □

Problem (5.4.4). *A continuous, adapted process $W = \{W_t, \mathcal{F}_t, 0 \le t < \infty\}$ is a d-dimensional Brownian motion if and only if*

$$f(W_t) - f(W_0) - \frac{1}{2}\int_0^t \Delta f(W_s)ds, \quad \mathcal{F}_t; 0 \le t < \infty$$

is in $\mathcal{M}^{c,loc}$ for every $f \in C^2(\mathbb{R}^d)$.

Proof. $\Longrightarrow$ is a direct result of Multidimensional Ito's Formula.

$\Longleftarrow$ Now suppose the defined process is a local martinagle for every $f \in C(\mathbb{R}^d)$, then in particular, it is a local martinagle when $f(x) = x_i$ and $f(x) = x_i x_j$ for $1 \le i,j \le d$. Then the desried result is a derict consequence of Levy's characterization of Brownian motion. □

Def (5.4.24). *A collection $\mathcal{D}$ of Borel- Measurable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ is called a determining class on $\mathbb{R}^d$ if for any two finite measures μ_1, μ_2 on $\mathcal{B}(\mathbb{R}^d)$, the identity*

$$\int_{\mathbb{R}^d} \varphi d\mu_1 = \int_{\mathbb{R}^d} \varphi d\mu_2$$

implies $\mu_1 = \mu_2$.

Problem (5.4.25). *Show that the collection $C_0^\infty(\mathbb{R}^d)$ is a determining class of $\mathbb{R}^d$.*

Proof. Direct consequence of Stone-Wairstrass or Riesz Representation. □

Supplementary Exercises

Excercise (5.4.33). *Assume that the coefficients $b_i : \mathbb{R}^d \to \mathbb{R}$ and $\sigma_{ij} : \mathbb{R}^d \to \mathbb{R}; 1 \le i \le d, 1 \le j \le r$ are measurable and bounded on compact subsets of $\mathbb{R}^d$, and let $\mathcal{A}$ be*

$$\mathcal{A}f(x) \triangleq \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d a_{lk}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i}$$

Let $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ be a continuous process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume the filtration satisfies the usual conditions. With $f \in C^2(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$, introduce the processes

$$M_t \triangleq f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds, \quad \mathcal{F}_t; \quad 0 \le t < \infty$$

$$\Lambda_t \triangleq e^{-\alpha t} f(X_t) - f(X_0) + \int_0^t e^{-\alpha s} \left(\alpha f(X_s) - \mathcal{A}f(X_s)\right) ds, \quad \mathcal{F}_t; \quad 0 \le t < \infty$$

and show that $M \in \mathcal{M}^{c,loc} \Leftrightarrow \Lambda \in \mathcal{M}^{c,loc}$. If f is bounded away from zero on compact sets and

$$N_t \triangleq f(X_t) \exp\left\{ - \int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)} ds \right\} - f(X_0), \quad \mathcal{F}_t; \quad 0 \le t < \infty$$

then thses two conditions are also equivalent to $N \in \mathcal{M}^{c,loc}$.

Proof. Assuming M is a local Martingale, then by the Integration by Parts formula from **Problem 3.3.12** we see that

$$\begin{aligned} e^{-\alpha t} f(X_t) &= f(X_0) + \int_0^t e^{-\alpha s} df(X_s) + \int_0^t f(X_s) de^{-\alpha s} \\ &= f(X_0) + \int_0^t e^{-\alpha s} dM_s + \int_0^t e^{-\alpha s} \mathcal{A}f(X_s)ds - \alpha \int_0^t e^{-\alpha s} f(X_s)ds \end{aligned}$$

rearranging terms to get the desired result. Now for the reverse direction,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t e^{\alpha s} d\left(e^{-\alpha s} f(X_s)\right) + \int_0^t e^{-\alpha s} f(X_s) de^{\alpha s} \\ &= f(X_0) + \int_0^t e^{\alpha s} d\Lambda_t - \int_0^t \alpha f(X_s) - \mathcal{A}f(X_s)ds + \alpha \int_0^t f(X_s)ds \end{aligned}$$

and rearranging the terms to get the desired result.

For the last equivalence relation, let's first note that the integral in $\exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)} ds\right\}$ is defined for almost all $\omega \in \Omega$ in Lebesgue sense, so this is a stochastic process with finite variation. With that said, let's apply integration by parts formula here while assuming M is a local martingale:

$$\begin{aligned} f(X_t) \exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)} ds\right\} &= \int_0^t \exp\left\{-\int_0^s \frac{\mathcal{A}f(X_u)}{f(X_u)} du\right\} df(X_s) - \int_0^t f(X_s) \frac{\mathcal{A}f(X_s)}{f(X_s)} \exp\left\{-\int_0^s \frac{\mathcal{A}f(X_u)}{f(X_u)} du\right\} ds + f(X_0) \\ &= \int_0^t \exp\left\{-\int_0^s \frac{\mathcal{A}f(X_u)}{f(X_u)} du\right\} dM_s + \int_0^t \exp\left\{-\int_0^s \frac{\mathcal{A}f(X_u)}{f(X_u)} du\right\} \mathcal{A}f(X_s)ds \\ &\quad - \int_0^t \mathcal{A}f(X_s) \exp\left\{-\int_0^s \frac{\mathcal{A}f(X_u)}{f(X_u)} du\right\} ds + f(X_0) \\ &= \int_0^t \exp\left\{-\int_0^s \frac{\mathcal{A}f(X_u)}{f(X_u)} du\right\} dM_s + f(X_0) \end{aligned}$$

The other direction is the same thing, so omit. □

Excercise (5.4.34). *Let* $(X, W), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ *be a weak solution to the funcitonal stochastic differential equation*

$$dX_t = b(t, X)dt + \sigma(t, X)dW_t$$

where we assume that

$$\|b(t, y)\| + \|\sigma_{i,j}\| \leq K_T; \quad 0 \leq t \leq T, \quad y \in C[0, \infty)^d$$

for all $0 < T < \infty$ *where* K_T *is a constant depending on* T. *For any continuous function* $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ *of class* $C^{1,2}((0, \infty) \times \mathbb{R}^d)$ *and any progressively measurable process* $\{k_t, \mathcal{F}_t, 0 \leq t < \infty\}$, *show that*

$$\Lambda_t \triangleq f(t, X_t) \exp\left\{-\int_0^t k_u du\right\} - f(0, X_0) - \int_0^t \left(\frac{\partial f}{\partial s} + \mathcal{A}'_s f - k_s f\right) \exp\left\{-\int_0^s k_u du\right\} ds, \mathcal{F}_t; \quad 0 \leq t < \infty$$

is in $\mathcal{M}^{c,loc}$. *If, furthermore, f and its indicated derivatives are bounded and k is bounded from below, then* Λ *is a martinagle.*

Proof. Let's assume we are working with the bounded case since for general case, we can define a stopping time such that the stopped process makes all those functions bounded, or simply put, localization.

Now, from **Problem 5.4.3** we see that

$$M_t^f \triangleq f(t, X_t) - f(0, X_0) - \int_0^t \left[\frac{\partial f}{\partial s} + \mathcal{A}'_s f\right](s, X_s)ds, \quad \mathcal{F}_t, \quad 0 \leq t < \infty$$

is a martinagle for all $f \in C^{1,2}((0, \infty) \times \mathbb{R}^d)$ that has bounded derivatives. Therefore, once again we can use the integration by part formula by realizing that $\exp\left\{-\int_0^t k_u du\right\}$ is a stochastic process with finite variation when k is bounded below, therefore,

$$\begin{aligned}
df(t, X_t) \exp\left\{-\int_0^t k_u du\right\} &= f(t, X_t) d \exp\left\{-\int_0^t k_u du\right\} + \exp\left\{-\int_0^t k_u du\right\} df(t, X_t) \\
&= -f(t, X_t)k_t \exp\left\{-\int_0^t k_u du\right\} dt + \exp\left\{-\int_0^t k_u du\right\} dM_t^f \\
&+ \exp\left\{-\int_0^t k_u du\right\} \left(\frac{\partial f}{\partial t} + \mathcal{A}'_t f\right)(t, X_t)dt
\end{aligned}$$

combine the terms we can see the desired result. □

Excercise (5.4.35). *Let the coefficients* b, σ *be bounded on compact subsets of* $\mathbb{R}^d$, *and assume that for each* $x \in \mathbb{R}^d$, *the time-homogeneous martinagle problem*

$$\mathbb{E}\left(f(y(t)) - f(y(s)) - \int_s^t (\mathcal{A}f)(y(u))du | \mathcal{B}_s\right) = 0$$

has a solution $\mathbb{P}^x$. *Suppose that there exists a function* $f : \mathbb{R}^d \to [0, \infty)$ *of class* $C^2(\mathbb{R}^d)$ *such that*

$$\mathcal{A}f(x) + \lambda f(x) \leq c, \quad \forall x \in \mathbb{R}^d$$

holds for some $\lambda > 0, c \geq 0$. *Then*

$$\mathbb{E}^x f(y(t)) \leq f(x)e^{-\lambda t} + \frac{c}{\lambda}(1 - e^{-\lambda t}); \quad 0 \leq t < \infty, x \in \mathbb{R}^d.$$

Proof. By assumption, under $\mathbb{P}^x$, the process $M_t^f \triangleq f(y(t)) - f(y(0)) - \int_0^t \mathcal{A}' f(y)ds$ is a martinagle with respect to the filtration given. Now take the expectation under $\mathbb{P}^x$ to get

$$\begin{aligned}
\mathbb{E}^x f(X_t) &= \mathbb{E}^x \left[f(X_0) + \int_0^t \mathcal{A}f(X_s)ds\right] \\
&\leq \mathbb{E}^x \left[f(X_0) + tc - \int_0^t \lambda f(X_s)ds\right] \\
&= f(x) + ct - \int_0^t \lambda \mathbb{E}^x [f(X_s)] ds
\end{aligned}$$

by Fubini since $f \geq 0$. Now apply Gronwall's inequality to obtain

$$\mathbb{E}^x f(X_t) \leq$$

□

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Problem. *this Consider $\mu, \delta \in \mathbb{R}$, and a standard one-dim Brownian Motion W and let $W_t^\mu = W_t + \mu t; 0 \le t < \infty$. Show that the process*

$$X_t = \int_0^t \exp\left[\delta\{W_t^\mu - W_s^\mu\} - \frac{1}{2}\delta^2(t-s)\right]$$

Satisfies the Shiryaev-Roberts stochastic integral equation

$$X_t = \int_0^t (1 + \delta\mu X_s)ds + \delta \int_0^t X_s dW_s.$$

Proof. Let $\eta_t = \delta W_t - \frac{1}{2}\delta^2 t + \delta\mu t$, then η is a semi-Mtg, and note that $\int_0^t \exp(\eta_s)ds$ is an increasing process. Therefore, we can use Ito's lemma, the multi-dim version. Set $f(x_1, x_2) = \exp(x_1)x_2$, and $X_t = f(\eta_t, \int_0^t \exp(-\eta_s)ds)$, so we have

$$\begin{aligned}
X_t = f(\eta_t, \int_0^t \eta_s ds) &= \int_0^t \exp(\eta_s) \int_0^s \exp(-\eta_u)du(\mu - \frac{1}{2}\delta)\delta ds + \int_0^t \exp(\eta_s - \eta_s)ds \\
&+ \frac{1}{2}\delta^2 \int_0^s \exp(\eta_s) \int_0^s \exp(\eta_u)duds + \int_0^t \exp(\eta_s) \int_0^s \exp(-\eta_u)dudW_s \\
&= \int_0^t \delta\mu X_s ds + t + \delta \int_0^t X_s dW_s
\end{aligned}$$

So we are done. □

Problem (Wald's Identity). *Let $\{B_s\}$ be the one-dim standard Brownian Motion, and let T be a stopping time. Show that if either*

(i) $\mathbb{E}[T] < \infty$

(ii) or $\{E[B_{t\wedge T}]\}$ is bounded in L^1

Then we have

$$\begin{aligned}
\mathbb{E}[B_T] &= 0 \\
\mathbb{E}[B_T^2] &= \mathbb{E}[T]
\end{aligned}$$

Proof. Note $\{B_{t\wedge T}\}$ is also a Mtg, and if (ii) is true, then this martingale is uniformly integrable, hence has a last element. Then by martingale convergence theorem, we have

$$\mathbb{E}[B_T] = \lim_{t\to\infty} \mathbb{E}[B_{t\wedge T}] = 0$$

So, we can can show that (i) implies (ii), then we have the first wald's identity. We can write B_T in the following form:

$$\begin{aligned}
|B_T| &= |\sum_{k=1}^{\lfloor T\rfloor} B_k - B_{k-1} + B_T - B_{\lfloor T\rfloor}| \\
&\le \sum_{k=1}^{\lfloor T\rfloor} |B_k - B_{k-1}| + |B_T - B_{\lfloor T\rfloor}| \\
&\le \sum_{k=1}^{\lfloor T\rfloor} \max_{0\le t\le 1} |B_{k-1+t} - B_{k-1}| + \max_{0\le t\le 1} |B_{\lfloor T\rfloor+t} - B_{\lfloor T\rfloor}| \\
&= \sum_{k=0}^{\lceil T\rceil} \max_{t\in[0,1]} |B_{k+t} - B_k| \\
&= \sum_0^\infty \mathbb{1}_{\lfloor T\rfloor \ge k} \max_{t\in[0,1]} |B_{k+t} - B_k|
\end{aligned}$$

We can do this since $T(\omega) < \infty$ on set of measure 1. For $T(\omega) = \infty$, then the floor and ceiling are just ∞ and we don't have the last terms in the first three lines.

Note also that $\max_{t\in[0,1]} |B_{k+t} - B_k| =_d \max_{t\in[0,1]} |B_t - B_0| = \max_{t\in[0,1]} |B_t|$, therefore, we have

$$\begin{aligned}
\mathbb{E}|B_T| &\leq \mathbb{E}[\sum_{k=0}^{\infty} \mathbb{1}_{T\geq k} \max_{t\in[0,1]} |B_{t+k} - B_k|] \\
&= \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{1}_{T\geq k} \max_{t\in[0,1]} |B_{t+k} - B_k|] \text{ Fubini, since positive} \\
&= \sum_{k=0}^{\infty} \mathbb{P}[T \geq k]\mathbb{E}[\max_{t\in[0,1]} |B_{t+k} - B_k|] \\
&= \sum_{k=0}^{\infty} \mathbb{P}[T \geq k]\mathbb{E}[\max_{t\in[0,1]} |B_t - B_0|] \\
&= \mathbb{E}[\max_{t\in[0,1]} |B_t - B_0|] \sum_{k=0}^{\infty} \mathbb{P}[T \geq k] \\
&= \mathbb{E}[\max_{t\in[0,1]} |B_t|]\mathbb{E}[T] < \infty
\end{aligned}$$

The finiteness is due to the identity $\mathbb{P}[\max_{0\leq t\leq T} B_t \geq a] = 2\mathbb{P}[B_t \geq a]$ and $\max_{0\leq t\leq T} B_t > 0$ a.s. and

$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}[X \geq a] dx$$

Note that $\mathbb{E}[|B_{t\wedge T}|]$ is bounded above by the above, hence it is uniformly integrable.

Now for the second identity. Let $M_t = B_t^2 - t$ is a Mtg, and $S_n = T_n \wedge T$ is a stopping time, where $T_n = \min_t\{|B_t| = n\}$. Therefore, $M_{t\wedge S_n}$ is also a mtg. Observe that

$$|B^2_{t\wedge S_n} - t \wedge S_n| \leq n^2 + T \text{ which is integrable}$$

So by Optional Sampling we have

$$\mathbb{E}[T_n \wedge T] = \mathbb{E}[B^2_{T_n\wedge T}]$$

Note that we have $T_n \wedge T \leq T$, Take the $n \to \infty$, the left side becomes $\mathbb{E}[T]$ by DCT. Now consider the following

$$\begin{aligned}
\mathbb{E}[B_T^2] &= \mathbb{E}[(B_T - B_{S_n} + B_{S_n})^2] \\
&= \mathbb{E}[(B_T - B_{S_n})^2] + 2\mathbb{E}[(B_T - B_{S_n})B_{S_n}] + \mathbb{E}[B^2_{S_n}]
\end{aligned}$$

Note that by Strong Markov Property, $(B_t - B_{S_n}) \perp S_{S_n}$ is another Brownian Motion. So by the first Wald's Identity, we have $\mathbb{E}[B_T - B_{S_n}] = 0$, hence the middle term is zero, therefore, we obtain the identity

$$\mathbb{E}[B_T^2] = \mathbb{E}[(B_T - B_{S_n})^2] + \mathbb{E}[B^2_{S_n}] \geq \mathbb{E}[B^2_{S_n}]$$

Therefore, $\lim_{n\to\infty} \mathbb{E}[B^2_{S_n}] \leq \mathbb{E}[B_T^2]$. Since $\lim_{n\to\infty} B_{T_n(\omega)\wedge T(\omega)}(\omega) = B_{T(\omega)}(\omega) a.e.$, then by Fatou's Lemma, we have

$$\mathbb{E}[B_T^2] \leq \liminf \mathbb{E}[B_{S_n}] = \liminf \mathbb{E}[T_n \wedge T] = \mathbb{E}[T]$$

By DCT or MCT. So we are done. □

Problem (3.3.35 More General Wald's Lemma)**.** *In the context of Wald's Lemma (problem above), but now under the condition $\mathbb{E}[\sqrt{T}] < \infty$, establish the Wald's Identity*

$$\mathbb{E}[W_T] = 0, \ \mathbb{E}[W_T^2] = \mathbb{E}[T]$$

Proof. By Mtg Moment Inequality, we have

$$\mathbb{E}[|B_{t\wedge T}|] \leq C\mathbb{E}[\sqrt{T \wedge t}] \leq C'\mathbb{E}[\sqrt{T}]$$

For some C' finite. Therefore, $|B_{t\wedge T}|$ is bounded in L^1, hence uniformly integrable. Hence we can use Mgt Convergence theorem to take $t \to \infty$, we we have the first identity.

For the second identity, note that $\mathbb{E}[\sqrt{T}] < \infty$ still gives the condition that $T < \infty$ a.s, so the argument above follows exactly the same. But for exercise, I'll try to reproduce the same argument:

Let T_n be the hitting time of $|B_t| = n$, hence $\lim_{n\to\infty} B_{T_n\wedge T} = B_T$ at least a.s. and consider

$$\begin{aligned}\mathbb{E}[B_T^2] &= \mathbb{E}[(B_T - B_{T_n\wedge T} + B_{T_n\wedge T})^2] \\ &= \mathbb{E}[(B_T - B_{T_n\wedge T})^2] + \mathbb{E}[(B_{T_n\wedge T})^2] + 2\mathbb{E}[(B_T - B_{T_n\wedge T})B_{T_n\wedge T})]\end{aligned}$$

By the strong Markov property, $B_{T_n\wedge T+t} - B_{T_n\wedge T}$ is an Brownian Motion that is independent of $B_{T_n\wedge T}$, and by the first Wald's identity, we have $\mathbb{E}[B_T - B_{T_n\wedge T}] = 0$, hence the last term vanishes. Therefore, we have $\mathbb{E}[B_T^2] = \mathbb{E}[(B_T - B_{T_n\wedge T})^2] + \mathbb{E}[(B_{T_n\wedge T})^2]$. So $\lim_{n\to\infty}\mathbb{E}[B^2_{T_n\wedge T}] \leq \mathbb{E}[B_T^2]$. However, by Fatou's Lemma we also have

$$\mathbb{E}[B_T] \leq \liminf \mathbb{E}[B_{T_n\wedge T}] = \liminf \mathbb{E}[T_n\wedge T] = \mathbb{E}[T]$$

The last equality is because $B^2_{T_n\wedge t} - T_n\wedge t$ is bounded in L^1, hence uniformly integrable + Mtg Convergence theorem. □

Problem (excercise 3.3.36 (M. Yor)). *Let R be a Bessel process with dim $d\geq 3$, starting at $r=0$. Show that $\{M_t \triangleq \frac{1}{R_t^{d-2}}; 1\leq t<\infty\}$*

(i) Is a local Mtg;

(ii) Satisfies $\sup_{1\leq t<\infty}\mathbb{E}[M_t^p] < \infty$ for all $0<p<\frac{d}{d-2}$ (and thus is uniformly integrable);

(iii) Is not a martingale.

Proof. (i) Need to find $\{T_n\}$ that increases to ∞ a.s. such that $M_{t\wedge T_n}$ are martingales for all n. Ito's lemma might be a good way to go, but R_t might be zero, so if those stopping time can bound it away from zero, then it might work. Let $f(x) = \frac{1}{\|x\|} = \frac{1}{(\sum_{k=1}^d x_k^2)^{\frac{d}{2}-1}}$ and let's x is bounded away from zero, then

$$\frac{\partial f}{\partial x_i} = \frac{(2-d)x_i}{\|x\|^d};\; f_{x_i,x_i} = (d-2)(d\frac{x_i^2}{\|x\|^{d+2}} - (d-2)\frac{1}{\|x\|^d})$$

So apply Ito's Lemma we have

$$\begin{aligned}M_t &= M_1 + \sum_{k=1}^d (2-d)\int_1^t \frac{W_s^k}{(\sum_{k=1}^d (W_s^k)^2)^{d/2}} dW_s^k + \frac{1}{2}\sum_{k=1}^d \int_1^t (d-2)(d\frac{(W_s^k)^2}{R_s^{d+2}} - (d-2)\frac{1}{R_s^d})ds \\ &= M_1 + \sum_{k=1}^d (2-d)\int_1^t \frac{W_s^k}{(\sum_{k=1}^d (W_s^k)^2)^{d/2}} dW_s^k\end{aligned}$$

which is definitely a mtg. So only need to find the stopping time, $T_n = \int_t\{M_t > \frac{1}{n}\}$ would work.

(ii) From the Mtg Moment Inequality, we have

$$\mathbb{E}[M_t^p] \leq \mathbb{E}[< M >_t^{\frac{p}{2}}] = (2-d)^{\frac{p}{2}}\mathbb{E}[\int_1^t \frac{1}{(\sum_{i=1}^d W_s^{(i)2})^{d-1}} ds]$$

□

Problem (5.4.33). *Assume $b_i, \sigma_{i,j} : \mathbb{R}^d \to \mathbb{R}$ for $1\leq i,j\leq d$ are measurable and bounded on compact sets of $\mathbb{R}^d$, and let $\mathcal{A}$ be associated operator*

$$\mathcal{A} \triangleq \frac{1}{2}\sum_{1\leq i,j\leq d} a^{i,j}\frac{\partial^2 f(x)}{\partial x_i\partial x_j} + \sum_{1\leq i\leq d} b^i(x)\frac{\partial f(x)}{\partial x_i}$$

Let $X = \{X_t, \mathcal{F}_t, 0\leq t<\infty\}$ be a continuous process on some prob space $(\Sigma, \mathcal{F}, \mathbb{P})$ and assume $\mathcal{F}_t$ satisfies the usual conditions. With $f\in C^2(\mathbb{R}^d)$ and $\alpha\in\mathbb{R}$, introduce the process

$$M_t \triangleq f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds, \mathcal{F}_t, 0\leq t<\infty$$

and

$$\Lambda_t \triangleq e^{-\alpha t}f(X_t) - f(X_0) - \int_0^t e^{-\alpha s}(\alpha f(X_s) - \mathcal{A}f(X_s)ds, \mathcal{F}_t, 0\leq t < \inf$$

and show that $M \in \mathcal{M}^{c,loc} \Leftrightarrow \Lambda \in \mathcal{M}^{c,loc}$. If f is bounded away from zero on compact sets, and

$$N_t \triangleq f(X_t)\exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}ds\right\} - f(X_0), \mathcal{F}_t, 0 \le t < \infty$$

then these two conditions are also equivalent to $N \in \mathcal{M}^{c,loc}$.(Hint: from the integration by parts formula we have if $M \in \mathcal{M}^{c,loc}$ and C_t is a continuous process of bounded variation, then $C_t M_t - \int_0^t M_s dC_s = \int_0^t C_s dM_s$ is in $M^{c,loc}$.)

Proof. First assume M_t be a local martingale, and let $C_t = e^{-\alpha t}$ and use the hint:

$$M_t C_t = e^{-\alpha t} f(X_t) - e^{-\alpha t} f(X_0) - e^{-\alpha t}\int_0^t \mathcal{A}f(X_s)ds$$

and

$$\int_0^t M_s dC_s = -\alpha \int_0^t e^{-\alpha s} f(X_s) - e^{-\alpha s} f(X_0) - e^{-\alpha s}\int_0^s \mathcal{A}f(X_u)duds$$

So we have a representation of the following martingale:

$$\int_0^t e^{-\alpha s} dM_s = e^{-\alpha t} f(X_t) - e^{-\alpha t} f(X_0) - e^{-\alpha t}\int_0^t \mathcal{A}f(X_s) + \alpha\left[e^{-\alpha s} f(X_s) - e^{-\alpha s} f(X_0) - e^{-\alpha s}\int_0^s \mathcal{A}f(X_u)du\right] ds$$

Here we assume WLOG that $f(X_0) = 0$, then

$$\int_0^t e^{-\alpha s} dM_s = e^{-\alpha t} f(X_t) - e^{-\alpha t}\int_0^t \mathcal{A}f(X_s)ds + \alpha\int_0^t e^{-\alpha s} f(X_s)ds - \alpha\int_0^t\int_0^s e^{-\alpha s}\mathcal{A}f(X_u)duds$$

Now consider the double integral

$$\begin{aligned}\alpha\int_0^t\int_0^s e^{-\alpha s}\mathcal{A}f(X_u)duds &= \alpha\int_0^t\int_u^t e^{-\alpha s}\mathcal{A}f(X_u)dsdu \\ &= \int_0^t e^{-\alpha u}\mathcal{A}f(X_u)du - \int_0^t e^{-\alpha t}\mathcal{A}f(X_u)du\end{aligned}$$

Now change u to s and replace the double integral we have $\int_0^t e^{-\alpha s} dM_s = \Lambda_t$, hence a local martingale.
Now let's say Λ_t is a local martingale, by assuming $f(X_0) = 0$ and letting $C_t = e^{\alpha t}$, we have:

$$\Lambda_t C_t = f(X_t) - e^{\alpha t}\int_0^t e^{-\alpha s}\alpha f(X_s) - e^{-\alpha s}\mathcal{A}f(X_s)ds$$

$$\int_0^t \Lambda_s dC_s = \alpha\int_0^t f(X_t) - e^{\alpha s}\int_0^s e^{-\alpha u}\alpha f(X_u) - e^{-\alpha u}\mathcal{A}f(X_u)duds$$

Again, consider the double integral

$$\begin{aligned}&\int_0^t\int_0^s \alpha e^{\alpha s - \alpha u}\left(\alpha f(X_u) - \mathcal{A}f(X_u)\right)duds \\ =&\int_0^t\int_u^t \alpha e^{\alpha s - \alpha u}\left(\alpha f(X_u) - \mathcal{A}f(X_u)\right)dsdu \\ =&e^{\alpha t}\int_0^t e^{-\alpha s}\left(\alpha f(X_s) - \mathcal{A}f(X_s)\right)ds - \int_0^t\left(\alpha f(X_s) - \mathcal{A}f(X_s)\right)ds\end{aligned}$$

But this back into the original expression, then we have

$$M_t = \Lambda_t C_t - \int_0^t \Lambda_s dC_t$$

which is also a local martingale. Note that the equivalent relation is true for all $\alpha \in \mathbb{R}$.
Now let $f(x) > \gamma_A > 0$ where $A \subseteq \mathbb{R}^d$ is compact.
Some observations:

$$d\exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}ds\right\} = -\exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}ds\right\}\frac{\mathcal{A}f(X_t)}{f(X_t)}dt$$

in Lebesgue's sense. For $\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}ds$, we can decompose the integrand into positive and negative parts, call them B^+, B^-. We can write the integral as $\int_0^t B_s^+ ds - \int_0^t B^- ds$, the difference between two pathwise nondecreasing processes, so derivative with respect to time makes sense. Now recall the integration by part formula: If $X_t = X_0 + M_t + B_t$ and $Y_t = Y_0 + N_t + C_t$, where $M, N \in \mathcal{M}^{c,loc}$ and B, C are continuous adapted processes with bounded variation with initial values zero, then we have

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - < M, N >_t$$

Note this also gives us a differential form for **Product Rule**, that is

$$d(X_s Y_s) = X_s dY_s + Y_s dX_s + d < M, N >_s$$

Now, let Y_t be the exponential term, then using the product rule, we have the following:

$$\begin{aligned}
d\left[f(X_t)Y_t\right] &= Y_t df(X_t) + f(X_t)dY_t \\
&= Y_t d\left(M_t + \int_0^t \mathcal{A}f(X_t)\right) - f(X_t)(\frac{\mathcal{A}f(X_t)}{f(X_t)}Y_t dt) \\
&= Y_t dM_t + Y_t \mathcal{A}f(X_t)dt - \mathcal{A}f(X_t)dt \\
&= Y_t dM_t
\end{aligned}$$

which gives a local Martingale. Now, suppose $N_t = f(X_t)\exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}ds\right\}$ be a local martingale. So the goal is to turn $M_t = f(X_t) - \int_0^t \mathcal{A}f(X_s)ds$ into some form of $Z_t dN_t$. We have

$$\begin{aligned}
dM_t &= df(X_t) - \mathcal{A}f(X_t)dt \\
&= dN_t \exp\left\{\int_0^t \frac{\mathcal{A}f(X_t)}{f(X_t)}\right\} - \mathcal{A}f(X_t)dt \\
&= \exp\left\{\int_0^t \frac{\mathcal{A}f(X_t)}{f(X_t)}\right\} dN_t + N_t \exp\left\{\int_0^t \frac{\mathcal{A}f(X_t)}{f(X_t)}\right\} \frac{\mathcal{A}f(X_t)}{f(X_t)}dt - \mathcal{A}f(X_t)dt \\
&= \exp\left\{\int_0^t \frac{\mathcal{A}f(X_t)}{f(X_t)}\right\} dN_t + f(X_t)\frac{\mathcal{A}f(X_t)}{f(X_t)}dt - \mathcal{A}f(X_t)dt \\
&= \exp\left\{\int_0^t \frac{\mathcal{A}f(X_t)}{f(X_t)}\right\} dN_t
\end{aligned}$$

Hence also a Martingale. □

Problem (5.4.34). *Let $(X, W), (\Omega, , \mathcal{F}, \mathbb{P}), \mathcal{F}_t$be a weak solution to the functional stochastic differential equation*

$$dX_t = b(t, X)dt + \sigma(t, X)dW$$

where condition $\|b(t,y)\| + \|\sigma_{i,j}(t,y)\| \leq K_T$ where $y \in C[0,\infty)^d$ hold for all $T \geq 0$, where K_T is a constant depending on T. For any continuous function $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ of class $C^{1,2}$ and any progressively measurable process $\{k_t, \mathcal{F}_t, 0 \leq t < \infty\}$, show that

$$\Lambda_t \triangleq f(t, X_t)\exp\left\{-\int_0^t k_u du\right\} - f(0, X_0) - \int_0^t \left(\frac{\partial f}{\partial s} + \mathcal{A}'_s f - k_s f\right)\exp^{-\int_0^s k_u du} ds$$

with $\mathcal{F}_t$ is in $\mathcal{M}^{c,loc}$. If, furthermore, f and its indicates derivatives are bouned and k is bouneded from below, then Λ is a martingale.

Proof. Again, assume WLOG that $f(0, X_0) = 0$. From problem 4.3 we have M_t^f is a martingale for all $f \in C^{1,2}$, where M_t^f is defined as follows:

$$M_t^f \triangleq f(t, X_t) - \int_0^t \left[\frac{\partial f}{\partial s} + \mathcal{A}'_s f(s, X_s)\right](s, X_s)ds$$

when k_t is progressively measurable (I am not sure why progressively measurable is important), as before, the exponential term has finite variation, where we have

$$d\Lambda_t = \exp\left\{-\int_0^t k_u du\right\} df(t, X_t) - f(t, X_t)\exp\left\{-\int_0^t k_u du\right\} k_t dt - \left(\frac{\partial f}{\partial t}(t, X_t) + \mathcal{A}'_t f(t, X_t) - k_t f(t, X_t)\right)\exp\left\{-\int_0^t k_s ds\right\} dt$$

Where $df(t, X_t) = dM_t^f + (\frac{\partial f}{\partial t}(t, X_t) + \mathcal{A}_t' f(t, X_t))dt$. Let's omit the arguments of functions for convinience, and let $Y_t = \exp\left\{-\int_0^t k_u du\right\}$. Therefore

$$\begin{aligned} d\Lambda_t &= Y_t dM_t^f + \left[Y_t \frac{\partial f}{\partial t} + Y_t \mathcal{A}_t' f\right] dt - f Y_t k_t dt - \frac{\partial f}{\partial t} Y_t - \mathcal{A}_t' f Y_t dt + k_t f Y_t dt \\ &= Y_t dM_t^f \end{aligned}$$

which is a local martingale. If we have the boundedness condition, then also by problem 5.4.3, we have M_t^f is a martingale, hence Λ_t is a martingale as well. □

Problem (5.4.35)**.** *Let the coefficients b, σ be bouned on compact subsets of $\mathbb{R}^d$, and assume that for each $x \in \mathbb{R}^d$, the time homogeneous martingale problem of Def 4.15 has a solution $\mathbb{P}^x$ satisfying (4,22). Suppoes that there exists a function $f : \mathbb{R}^d \to [0, \infty)$ of class $C^2(\mathbb{R}^d)$ such that*

$$\mathcal{A}f(x) + \lambda f(x) \le c, \forall x \in \mathbb{R}^d$$

holds for some $\lambda > 0$ and $c \ge 0$. Then

$$E^x[f(y(t))] \le f(x)e^{-\lambda t} + \frac{c}{\lambda}(1 - e^{-\lambda t});\ 0 \le t < \infty, x \in \mathbb{R}^d$$

Proof. Being a solution of time homogeneous martingale problem, we have

$$\mathbb{E}\left[f(y(t)) - f(y(s)) - \int_s^t \mathcal{A}f(y(u))du | \mathcal{B}_s\right] = 0\ \mathbb{P}a.s.$$

and $\mathbb{P}^x[y(0) = x] = 1$. So $f(X_t) - \int_0^t \mathcal{A}f(X_s)ds$ is a martingale where X_t is the coordinate mapping process of continuous function in $\mathbb{R}^d$. Assume WLOG that $f(x) = 0$, and denote $\mathbb{E}^x[f(X_t)]$ as Z_t, then we have the following

$$\begin{aligned} Z_t &= \mathbb{E}[M_t^f + \int_0^t \mathcal{A}f(X_s)ds] \\ &= \mathbb{E}[\int_0^t \mathcal{A}f(X_s)ds] \\ &\le \mathbb{E}[\int_0^t ct - \lambda f(X_s)ds] \\ &= ct - \lambda \int_0^t Z_s ds \text{ Fubini's since } f \ge 0 \end{aligned}$$

Then by Gronwall's we have

$$Z_t \le cte^{-\lambda t} \le \frac{c}{\lambda}(1 - e^{-\lambda t})$$

Not sure how to prove the last inequality, but it is true. □

Problem (5.7.3)**.** *Let $\mathcal{A}$ be elliptic in the open bounded domain D, and $k, g : \overline{D} \to \mathbb{R}$ and $f : \partial D \to \mathbb{R}$ Let u be the solution fo the Dirichlet problem:*

$$\begin{cases} \mathcal{A}u - ku &= -g;\ in\ D \\ u &= f;\ on\ \partial D \end{cases}$$

Let $\tau_D \triangleq \inf\{t \ge 0; X_t \notin D\}$. If

$$E^x \tau_D < \infty; \forall x \in D$$

Show that under (7.2)-(7.4), we have

$$u(x) = \mathbb{E}^x\left[f(X_{\tau_D}) \exp\left\{-\int_0^{\tau_D} k(X_s)ds\right\} + \int_0^{\tau_D} g(X_t) exp\left\{-\int_0^{\tau_D} k(X_s)ds\right\} dt\right]$$

for every $x \in \overline{D}$.

Proof. First, $X_t, \mathbb{P}, \mathcal{F}_t$ is a weaks solution of

$$X_s^{(t,x)} = x + \int_t^s b(X_s^{(t,x)\theta})d\theta + \int_t^s \sigma(X_s^{(t,x)})dW_\theta$$

I think this problem assumes $t = 0$, hence its just a usual SDE, hence it omit a class of martingales for any $u \in C^{1,2}$:

$$M_t^u = u(X_t) - u(x) - \int_0^t \mathcal{A}u(X_s)ds$$

since $\mathbb{P}(X_0) = x$ a.s. Hence

$$du(X_t) = M_t^u + \int_0^t \mathcal{A}u(X_s)ds$$

this is due to Prop 4.11. Therefore, by the integration by parts formula we have:

$$\begin{aligned} du(X_t)\exp\left\{-\int_0^t k(X_s)\right\} &= \exp\left\{-\int_0^t k(X_s)\right\} du(X_t) - k(X_t)\exp\left\{-\int_0^t k(X_s)\right\} u(X_t)dt \\ &= \exp\left\{-\int_0^t k(X_s)\right\} dM_t^u + \exp\left\{-\int_0^t k(X_s)\right\} [\mathcal{A}u(X_t)dt - k(X_t)u(X_t)]\, dt \\ &= \exp\left\{-\int_0^t k(X_s)\right\} dM_t^u - \exp\left\{-\int_0^t k(X_s)\right\} g(X_t)dt \end{aligned}$$

Integrate both sides from 0 to $T \wedge \tau$ and take expectation to get

$$\mathbb{E}^x\left[u(X_{T\wedge\tau})\exp\left\{-\int_0^{T\wedge\tau} k(X_s)\right\}\right] - u(x) = \mathbb{E}\left[-\int_0^{T\wedge\tau} \exp\left\{-\int_0^t k(X_s)\right\} g(X_s)ds\right]$$

By bounded convergence theorem, take $T \to \infty$ to get the desired result. □

Problem (5.7.7). *In the case of bounded coefficients, i.e.*

$$|b_i(t,x)| + \sum_{j=1}^r \sigma_{i,j}^2(t,x) \le \rho, 0 \le t < \infty, x \in \mathbb{R}^d, 1 \le i \le d$$

Show that the polynomial condition (7.14) in Theorem 7.6 may be replaced by

$$\max_{0\le t\le T} |v(t,x)| \le Me^{\mu\|x\|^2}, x \in \mathbb{R}^d$$

for some $M > 0$ *and* $0 < \mu < \frac{1}{18}\rho Td$ *(Hint: Use problem 3.4.12)*

Proof. Hint says if $X_t = x + M_t + C_t$ where $M_t \in \mathcal{M}^{c,loc}$, C_t a continuous process with bounded variation, and if $|C_t| + < M_t > \le \rho t$, then for fixed T and large n we have

$$\mathbb{P}\left[\max_{0\le t\le T} |X_t| \ge n\right] \le \exp\left\{\frac{-n^2}{18\rho T}\right\}$$

so the proof is exactly the same, but use the result of the hint problem instead of Chebyshev's inequality. □

I am done with the first read!